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Theme

Averaged control and no-regret control to study some distributed systems with missing data.

Presented by:
Mouna Abdelli
Before the jury:

Mr. Elhadj Zeraoulia	Professor	University of Tebessa	President
Mr. Faycal Merghadi	MCA	University of Tebessa	Supervisor
Mr. Abdelhak Hafdallah	MCA	University of Tebessa	Co-Supervisor
Mr. Abdelhamid Ayadi	Professor	University of O.E.B	Examiner
Mr. Nouri Boumaza	MCA	University of Tebessa	Examiner
Ms. Amel Berhail	MCA	University of Guelma	Examiner

Defense date: ..../....../....

Contents

To My Father Who Never Saw This Success
In Loving Memory.
To My Mother
With Love and Eternal Appreciation
To My Brothers and Sisters

"Mathematics knows no races or geographic boundaries,
for mathematics, the cultural world is one country"
David Hilbert

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Abstract

The issue addressed in this thesis concerns the averaged controllability of some distributed systems with missing data or depending on a parameter. First, we studied the averaged null controllability problem by the Hilbert Uniqueness Method for some parameter dependent hyperbolic systems. Where we proved some uniqueness theorems which extend those Lions [43] to the case of parameter dependent wave equation and those Zuazua [70] to the case of parameter dependent vibrating plate equation, these results may often be got by using the multiplier method. Then, we gave the averaged null controllability results. The used main tool is the notion of averaged control which has been introduced recently by Zuazua [69]. Afterward, we discussed an extension of an averaged controllability problem of parameter dependent hyperbolic systems to the case where the target defined only in a part of the system domain. We presented a definition and some properties of the regional averaged controllability notion and after that, we characterized the control which achieving the regional averaged controllability with minimum energy. Second, we studied general and abstract control systems with missing data. By using both the averaged control notion and the no-regret method introduced for the optimal control of systems with missing data, we introduced the notions of averaged no-regret control and its approximation to get a full characterization for the optimal control. As an example, we apply the described theory on a parameter dependent electromagnetic wave equation with missing initial conditions.

Key Words:

Averaged control, distributed systems, regional averaged controllability, averaged no-regret control, missing data, unknown parameters.

Résumé

La problématique abordée dans cette thèse concerne la contrôlabilité moyenne de certains systèmes distribués avec des données manquantes ou dépendant d'un paramètre. Tout d'abord, nous avons étudié le problème de la contrôlabilité nulle moyenne par la méthode d'unicité de Hilbert de certains systèmes hyperboliques dépendant d'un paramètre. Où nous avons prouvé certains théorèmes d'unicité qui étendent ceux [43] au cas de l'équation d'onde dépendant d'un paramètre et ceux Zuazua [70] au cas de l'équation de la plaque vibrante dépendant d'un paramètre, ces résultats peuvent souvent être obtenus en utilisant la méthode du multiplicateur. Ensuite, nous avons donné les résultats de contrôlabilité nulle moyenne. L'outil principal utilisé est la notion de contrôle moyenné qui a été introduite récemment par Zuazua [69]. Puis, nous avons discuté une extension d'un problème de contrôlabilité moyenne des systèmes hyperboliques dépendants de paramètres au cas où la cible est définie seulement dans une partie du domaine du système. Nous avons présenté une définition et quelques propriétés de la notion de contrôlabilité moyenne régionale et après cela, nous avons caractérisé le contrôle qui permet d'atteindre la contrôlabilité moyenne régionale avec une énergie minimale. Deuxièmement, nous avons étudié des systèmes de contrôle généraux et abstraits avec des données manquantes. En utilisant à la fois la notion de contrôle moyen et la méthode sans regret introduite pour le contrôle optimal des systèmes avec des données manquantes, nous avons introduit les notions de contrôle moyen sans regret et son approximation pour obtenir une caractérisation complète pour le contrôle optimal. Comme exemple, nous appliquons la théorie décrite sur une équation d'onde électromagnétique dépendant d'un paramètre avec des conditions initiales manquantes.

Mots clés:

Control moyen, systèmes distribués, contrôlabilité moyenne régionale, control sans regret, données

manquantes, paramètres inconnu.

ملخص

تتعلق القضية التي تم تناولها في هذه الرسالة بالقدرة على التحكم المتوسط لبعض الأنظمة التوزيعية ذات البيانات المفقودة أو المرتبطة بوسيط. أولاً، درسنا مشكلة قابلية التحكم الصفرية المتوسطة بواسطة طريقة الوحدانية الهيلبيرتية لبعض الأنظمة الزائدية المرتبطة بوسيط. حيث أثبتنا بعض نظريات الوحدانية التي توسع نظريات ليونس ﴿٨﴾ إلى حالة معادلة الموجة المرتبطة بوسيط و نظريات زوازوا ﴿٦٩﴾ إلى حالة معادلة لوحة الاهتزاز المرتبطة بوسيط، هذه النتائج غالبا ما يمكن الحصول عليها بإستخدام طريقة المضاعف. فيما بعد، أعطينا نتائج قابلية التحكم الصفرية المتوسطة. الأداة الرئيسية المستخدمة هي مفهوم التحكم المتوسط الذي تم تقديمه مؤخراً بواسطة زوازوا ﴿٦٨﴾. بعد ذلك، ناقشنا امتداد لمشكلة قابلية التحكم المتوسطة للأنظمة الزائدية المرتبطة بوسيط إلى حالة لما يكون الهدف معرف على جزء فقط من مجال النظام. قدمنا تعريفا و بعض الخصائص لمفهوم قابليلة التحكم المتوسطة الجهوية و من ثم قمنا بتوصيف التحكم الذي يحقق قابلية التحكم المتوسطة الجهوية بأقل قدر من الطاقة. ثانيا، درسنا أنظمة التحكم العامة والمجردة ذات البيانات المفقودة. باستخدام كل من مفهوم التحكم المتوسط وطريقة عدم الندم المقدمة للتحكم الأمثل في الأنظمة ذات البيانات المفقودة، قدمنا مفاهيم متوسط التحكم في عدم الندم وتقريبه للحصول على توصيف كامل للتحكم الأمثل. كمثال، نطبق النظرية الموصوفة على معادلة الموجة الكهرومغناطيسية المرتبطة بوسيط ذات شروط أولية مفقودة.

الكلمات المفتاحية:

التحكم المتوسط، الأنظمة التوزيعية، قابلية التحكم المتوسطة الجهوية، التحكم المتوسط دون ندم، بيانات مفقودة، وسيط مجهول.

List of Figures

List of scientific activities

List of publications

1. M. Abdelli, A. Hafdallah, F. Merghadi and M. Louafi. Regional averaged controllability for hyperbolic parameter dependent systems. Control Theory Tech, Vol. 18, No. 3, pp. 307–314, 2020. DOI https://doi.org/10.1007/s11768-020-0006-5.
2. M. Abdelli and A. Hafdallah. Averaged null controllability for some hyperbolic equations depending on a parameter. Journal of Mathematical Analysis and Applications, Volume 495, Issue 1, 2021, https://doi.org/10.1016/j.jmaa.2020.124697.
3. A. Hafdallah and M. Abdelli . Averaged No-Regret Control for an Electromagnetic Wave Equation Depending upon a Parameter with Incomplete Initial Conditions. Electromagnetic Wave Propagation for Industry and Biomedical Applications. IntechOpen, 2021, DOI: 10.5772/intechopen.95447.

Un published papers

1. M. Abdelli and C. Castro. Numerical approximation of the averaged controllability for the wave equation with unknown velocity of propagation, 2020, arXiv:2010.08746v1.

Conference presentations

1. M. Abdelli, Regional gradient controllability of hyperbolic systems. Talk presented at Workshop Trends in Partial Differential Equations and Related Fields 2019. Sidi Bel-Abbes University.

2. M. Abdelli and A. Hafdallah, Regional controllability with minimum energy of wave equation. Talk presented at Identification and control: some challenges 2019. Monastir University.

3. M. Louafi, A. Hafdallah and M. Abdelli, About averaged controllability of systems depending on a parameter. Talk presented at Identification and control: some challenges 2019. Monastir University.

4. M. Abdelli and A.Hafdallah, F. Merghadi. About average control, Talk presented at National Day on Applied Mathematics (JNMA'2019). Oum el Bouaghi University.

Notations and abbreviations

$\partial\Omega = \Gamma$: Boundary of Ω.

$\|.\|_{\mathcal{V}}$: A norm in Banach space $\mathcal{V}$.

$|.|_{\mathcal{V}}$: A semi-norm in $\mathcal{V}$.

$(.,.)_{\mathcal{V}}$: A scalar product in Hilbert space $\mathcal{V}$.

$(.,.)_{\mathcal{V},\mathcal{V}'}$: Duality productbetween $\mathcal{V}$ and $\mathcal{V}'$.

$\mathrm{Im}(\mathcal{V})$: Image of $\mathcal{V}$.

$\ker(\mathcal{V})$: Kernel of $\mathcal{V}$.

lim : Limit.

$\nabla = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right)^T$: The gradient operator.

$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$: The laplacien operator.

$\frac{\partial y}{\partial \eta} = \nabla y.\eta$: The conormal derivative.

div : Divergence.

/ : Such as.

A^* : The adjoint operator of A.

$d\Gamma$: Lebesgue measure on boundary Γ.

χ_ω : Characteristic function of the set ω.

a.e: Almost everywhere.

sup : Supremum.

inf : Infimum.

$\mathcal{L}(\mathcal{V}, \mathcal{Z})$: The space of linear bounded operators from $\mathcal{V}$ to $\mathcal{Z}$.

$C^2(\Omega)$: The class of functions with continuous first and second derivative on Ω.

$C^k(\Omega)$: space of the functions k times continuously differentiable on $\Omega, k \geq 0$.

$\mathcal{D}(Q)$: The space of functions in C^∞ with a compact support in Q.

$\mathcal{D}'(Q)$: The dual space of $\mathcal{D}(Q)$.

$L^2([0,T], U)$: Space of L^2-integrable functions from $[0,T]$ to U.

$L^P(\Omega)$: $\{$measurable on Ω and $\int_\Omega |f(y)|^p dy < \infty\}, 1 \leq p < \infty$.

$L^\infty(\Omega)$: $\{$measurable on Ω and and there exists $c > 0 : |f(y)| \leq c$, a.e on $\Omega\}$.

$H^1(\Omega) : \{u \in L^2(\Omega) : \dfrac{\partial u}{\partial x_i} \in L^2(\Omega), \forall i = 1, \ldots, n\}$: Sobolev space of order 1.

$H_0^1(\Omega) : \{u \in H^1(\Omega) : u = 0$ on $\partial\Omega\}$.

$H^m(\Omega) : \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega), \forall \alpha \in \mathbb{N}^n, |\alpha| \leq m\}$:Sobolev space of order m.

$W^{m,p}(\Omega) : \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), \forall \alpha \in \mathbb{N}^n, |\alpha| \leq m\}$:Sobolev space of order m.

$\mathcal{V}'$: Dual of $\mathcal{V}$.

a.e.: Almost every where.

PDE: Partial differential equations.

ODE: Ordinary differential equations.

$\rightharpoonup$: Symbol of weak convergence.

iff: If and only if.

w.r.t.: With respect to.

s.t.: Such that.

Introduction

Control theory is an interdisciplinary branch of applied mathematics that deals with the behavior of evolution systems modeled by partial or ordinary differential equations (PDE/ODE). It starts to emerge in 1960 but, it is difficult to tell where the first mathematical formulation of the problem literally originates, one can say that it was introduced by Kalman and Lions [28], [41], [38]. Control theory has obviously great potential in the sense of various applications in different fields: physics, biology, mechanical engineering, astronomy, social science, medicine [24], [58]. Hence, it is very well known.

The general problem of control theory amounts to choosing a certain control that would lead a system from the given initial state to the prescribed final state. This type of problem is known as the controllability problem which played an important role through the history of control theory. Many fundamental problems of control theory (stability and stabilization, optimal control) can only be solved under the assumption that the system is controllable. For this reason, it has been studied by several authors for finite dimensional systems, modeled by ODEs [62], [47] and for infinite dimensional systems, modeled by PDEs (distributed parameter systems) [41], [43].

In the last three decades, the regional controllability concept is commonly applied in practical applications, particularly the possibility of steering the initial state to a prescribed state defined only in a given subregion $\omega \in \Omega$. For example, in an industrial furnace, it may be that the control is only required to maintain the temperature at a certain level in a prescribed subregion of the furnace. This problem has been introduced and developed by El Jai et all. in [13], [12], [13], [68]. A new direction in distributed systems has been whether a system not controllable on the global domain Ω can be controllable on a region ω of that domain Ω.

Due to the uncertainties and complexities of the modeling of physical processes, dynamic population and many other specialties, it is difficult to perfectly model it, thus it becomes natural to model them using parameter dependent system which is a system whose dynamics are governed by parameter dependent operators [63]. In such a situation, we cannot require exact controllability of the system by a single control (using a control independent of the parameter). To avoid this paradox, Zuazua introduced the notion of averaged controllability in [69]. There, the problem was introduced and solved in the setting of finite-dimensional systems, and afterwards generalized to PDE setting in [35], [48], [49]. Its goal is to control the averaged of system components instead of the state with respect to the unknown parameter.

Moreover, modeling those problems may also lead to mathematical systems with missing data (missing initial conditions, missing second member, and possibly missing boundary conditions). For example, we never know the initial data in almost all problems of meteorology, oceanography or the problems of pollution in a river. We have a wide range of options when it comes to the option of the initial moment. Hence, we never know the initial data. Furthermore, in biomedical, boundary conditions may also be unknown, which may, for instance, be unavailable for measurements. Generally speaking, in order to control such systems, we use the notion of *no-regret control*. This notion was introduced many years later in statistics by Savage [61]. Then, Lions has applied this notion and another called *low-regret control* to problems with incomplete data in several works [40], [17], [44], [42] for different applications. Since then, several scholars have used the idea of these concepts to control systems with missing data or with incomplete data. This concept was applied later by Nakoulima et al. [57] to control distributed linear systems with missing data. Thereafter, they developed their studies to control some nonlinear distributed systems with incomplete data [54]. In [56], Omrane et al. applied the notion of *low-regret control* of an ill-posed backward heat equation. Jacob and Omrane [25] generalized the notion of *no-regret control* to a linear age-structured population dynamics of incomplete initial data. The authors in all those works proved the uniqueness of the *low-regret control* and converge to the *no-regret control* for which they obtained a singular optimality system.

In recent years, an interesting notion has been introduced to control a parameter dependent system with missing boundary conditions which is *averaged no-regret control*. In [20], Hafdallah and Ayadi was introduced this concept by combining the notion of *averaged control* and the *no-regret control* notion to control an electromagnetic wave displacement depending on unknown velocity of propagation and with missing Dirichlet boundary condition. Subsequently, Mophou [52] also used this new concept to control the general heat governed by an operator depending on an unknown parameter and with missing boundary conditions. Then, Hafdallah [21] generalized the notion of *averaged no-regret control* to an abstract control system with incomplete data.

An Overview of the Thesis

In this thesis, we focus on the control problem of distributed systems (particularly distributed hyperbolic systems) depending on an unknown parameter or with missing data using the notions of *averaged control* and *no- regret control*. There are a total three chapters, the chapter wise description is given below

First chapter [**1**], mainly contains some useful ingredients related to control theory that will be used to obtain our main results. More precisely, we started with a short overview on exact, weak and regional controllability with some necessary theorems. Then, we briefly discuss the classical theory of optimal control for distributed systems (which has been introduced first by Lions in [45]). Next, we introduce the notion of averaged control by giving different notions of averaged controllability (exact, weak, null) with their required characterizations. At the end, we present a short description of no-regrets control, low-regret control for stationary problems.

Second chapter [**2**], this chapter is divided into three parts. The first part is devoted to studying the problem of averaged controllability for parameter dependent wave equation. This study is an extension of Lions approach [43] to the case of parameter dependent system. We prove some uniqueness theorems and we give the null averaged controllability results. In the second part, we use the same steps used in the previous section to treat the problem of vibrating plate equation i.e. we demonstrate an averaged inverse and direct inequalities giving some coercivity and continuity results for the main introduced operator in Hilbert uniqueness method and moreover, we describe the main steps of the Hilbert uniqueness method for the averaged null controllability problem. The third part is dedicated to the regional averaged controllability of parameter dependent hyperbolic systems of internal zone actuator and internal pointwise actuator. The focus is on the characterization of the control achieving the regional averaged controllability with minimum energy. The approach is based on the Hilbert Uniqueness Method.

Third chapter [**3**], we study general and abstract control systems depending on a parameter and with missing data. By combining the low-regret technique and the averaged control notion. As an example,we study the optimal control problem for an electromagnetic wave equation with a potential term depending on a real parameter and with missing initial conditions.

Finally, we conclude the work carried out in this thesis and indicate interesting directions for future work. At the end of the thesis, we include several classical results that are used throughout the thesis in the Appendix.

Chapter 1

Background and preliminaries

In this chapter, we briefly describe the mathematics background of the control theory by considering the linear case of distributed parameter system which are widely used in the formulations and proofs of the major results. An overview of the concepts used in this thesis was given such as controllability, regional controllability optimal control, averaged control, Pareto control, no-regret control, and low-regret control.

1.1 Controllability of distributed systems

Distributed parameter systems are systems in which the variables depend on time and space. They are defined by operators (A, B, C) where A defines the dynamics of the system, B is the control operator describing the inputs and C is the observation operator in terms of outputs. They are usually described by partial differential equations that can be linear or nonlinear, discrete or continuous, deterministic or stochastic.

Controllability plays an important role. Roughly speaking, controllability in general means that it is possible to steer a dynamic system from an initial state to an desired state using a set of admisible controls.

In this section, we give some classical results about controllability and regional controllability of distributed parameter systems in infinite-dimensional spaces.

1.1.1 Problem statement

Consider the system described by the state equation

$$\begin{cases} \dfrac{dy}{dt}(t) = Ay(t) + Bu(t), \ \forall t \in]0, T[, \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where the state space is $\mathcal{V}$, $A \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$, U is the control space, $B \in \mathcal{L}(U, \mathcal{V})$ is the control operator, $u \in L^2([0,T], U)$ is the control function.
We suppose that the operator A generates a strongly continuous semi-group $\{S(t)\}_{t\geq 0}$ (See Appendix). The semi-group plays a prominent role in the determination of the solution of an abstract differential equation. Particularly the solution of the system (1.1) is given by

$$y(t) = S(t)y_0 + \int_0^t S(t-\tau)Bu(\tau)\,d\tau \quad ,t \in [0,T]. \tag{1.2}$$

1.1.2 Exact & weak controllability

We denote by y_u the solution of the system (1.1) excited by the control u and then take the variable space Ω is a domain of $\mathbb{R}^n$ with a regular boundary Γ, $\mathcal{V} = L^2(\Omega)$ and $U = \mathbb{R}^p$.
The formulation of the controllability problem of the system (1.1) is as follows:
Given a time $T > 0$ and an initial suitable condition y_0, is there a control u such that the solution $y = y_u(t)$ satisfies the condition $y(T) = y_d$? where $y_d \in \mathcal{V}$ is a desired state priori chosen.
In other words: Study the existence of a control u which steer the system to the desired state y_d at time $T > 0$.
Now, we introduce some notions of exact and weak controllability.

Definition 1.1 [4] *The system (1.1) is said to be exactly controllable on $[0,T]$ if for every final target y_d in $\mathcal{V}$, there exists a control $u \in L^2([0,T], U)$ such that*

$$y(T) = y_d. \tag{1.3}$$

*This notion is illustrated on **Figure 1**.*

Figure 1: *Exact controllability.*

Definition 1.2 *[4] The system (1.1) is said to be weakly controllable on $[0, T]$ if for every final target y_d in $\mathcal{V}$ and for all $\varepsilon > 0$, there exists a control $u \in L^2([0, T], U)$ such that*

$$\|y(T) - y_d\|_{\mathcal{V}} \leq \varepsilon. \tag{1.4}$$

*This notion is illustrated on **Figure 2**.*

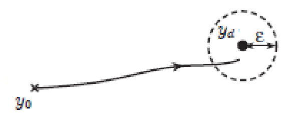

Figure2: *Weak controllability.*

1.1.3 The controllability operator

Let $\mathcal{H}$ be the operator defined in $L^2([0, T]\,; U)$ with its values taken in $\mathcal{V}$

$$\begin{aligned} \mathcal{H} \quad &: \quad L^2([0, T]; U) \rightarrow \mathcal{V}, \\ u \quad &\mapsto \quad \mathcal{H}u = \int_0^T S(T - \tau)Bu\,(\tau)\,d\tau. \end{aligned} \tag{1.5}$$

Proposition 1.1 *[15] The system (1.1) is said to be exactly controllable on $[0, T]$ iff*

$$\forall y^* \in \mathcal{V}', \exists \gamma > 0 : \|B^* S^*\,(.)\,y^*\|_{L^2([0,T],U)} \geq \gamma\,\|y^*\|_{\mathcal{V}'}. \tag{1.6}$$

We introduce the following matrix

$$\mathcal{G} = \mathcal{H}\mathcal{H}^* = \int_0^T S\,(T - t, \sigma)\,B\,(\sigma)\,S^*\,(T - t, \sigma)\,B^*\,(\sigma)\,dt, \tag{1.7}$$

which is the **controllability Gramian** where S^*, B^* are the adjoint of S, B repectivly.

1.1.4 Controllability and actuators

Definition 1.3 *[15] An actuator is defined by a couple (D, f), where D is the support of the actuator and f is the spatial distribution of the action on the support D.*

Remark 1.1 *In the case of pointwise actuator (internal or boundary), D is reduced to the location $b \in \Omega$ and $f = \delta(. - b)$ where δ is the dirac mass.*

We assume that the system (1.1) is excited by p actuators $(D_i, f_i)_{1\le i\le p}$ such that $D_i \cap D_j = \emptyset$ if $i \neq j$.

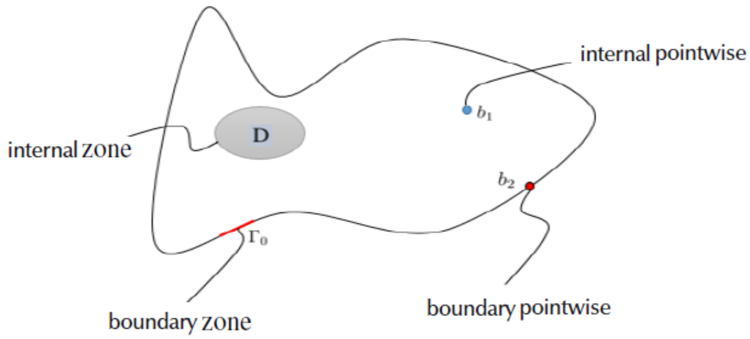

Figure 3: Some types of actuator supports.

Proposition 1.2 *[15] A sequence of actuators $(D_i, f_i)_{1\le i\le p}$ are said to be strategic, if the system excited by these actuators is weakly controllable.*

Remark 1.2 *In applications, dynamic systems that are controllable over all the domain are rare, hence the necessity to study this concept only on part of the domain. For this, we define the notion of regional controllability.*

1.1.5 Regional controllability

For regional controllability, we want the state of the system at time T verify the desired property on part of the domain. Let a subdomain (a region) ω of Ω suppose not empty. We consider the restriction function

$$\begin{aligned} \chi_\omega \quad &: \quad L^2(\Omega) \to L^2(\omega), \\ y \quad &\mapsto \quad \chi_\omega(y) = y|_\omega. \end{aligned}$$

Whose the adjoint $\chi_\omega^* : L^2(\omega) \to L^2(\Omega)$ is

$$(\chi_\omega^* y)(x) = \begin{cases} y(x) & \text{si } x \in \omega, \\ 0 & \text{si } x \in \Omega \backslash \omega. \end{cases}$$

Definition 1.4 *[4] The system (1.1) is said to be exactly regionally controllable on ω if, for every final target $y_d \in L^2(\omega)$, there exists a control $u \in L^2([0,T], U)$ such that*

$$y_u(T)|_\omega = y_d. \tag{1.8}$$

Definition 1.5 *[4] The system (1.1) is said to be weakly regionally controllable on ω if, for every final target $y_d \in L^2(\omega)$, there exists a control $u \in L^2([0,T],U)$ such that*

$$\|y_u(T)|_\omega - y_d\|_{L^2(\omega)} \leq \varepsilon. \tag{1.9}$$

Proposition 1.3 *[15]*

i) The system (1.1) is said to be exactly regionally controllable on ω iff

$$\operatorname{Im} \chi_\omega \mathcal{H} = L^2(\omega). \tag{1.10}$$

ii) The system (1.1) is said to be weakly regionally controllable on ω iff

$$\overline{\operatorname{Im} \chi_\omega \mathcal{H}} = L^2(\omega). \tag{1.11}$$

Remark 1.3

i) A system that is exactly (respectively weakly) controllable is exactly (respectively weakly) regionally controllable on any region ω from Ω.

ii) If ω_1 and ω_2 are two regions such that $\omega_2 \subset \omega_1$, then a system that is exactly (respectively weakly) regionally controllable on ω_1 is exactly (respectively weakly) controllable on ω_2.

Proposition 1.4 *[15] The system (1.1) is said to be exactly regionally controllable on $[0,T]$ iff*

$$\forall y^* \in L^2(\omega), \exists \gamma > 0 : \|B^* S^*(.)\, y^*\|_{L^2([0,T],U)} \geq \gamma \|y^*\|_{L^2(\omega)}. \tag{1.12}$$

Proposition 1.5 *[66]*

i) The system (1.1) is exactly regionally controllable on ω iff

$$\ker \chi_\omega + \operatorname{Im} \mathcal{H} = L^2(\Omega). \tag{1.13}$$

ii) The system (1.1) is said to be weakly regionally controllable on ω iff

$$\ker \chi_\omega + \overline{\operatorname{Im} \mathcal{H}} = L^2(\Omega). \tag{1.14}$$

In the case where A generates a strongly continuous analytic semi-group $\{S(t)\}_{t\geq 0}$ (See Appendix), then we have the following result

Proposition 1.6 *[66] The system (1.1) is weakly regionally controllable on ω iff*

$$\overline{\bigcup_{n\geq 0} \operatorname{Im}(\chi_\omega A^n S(t) B)} = L^2(\omega), \ \forall t \in [0,T]. \tag{1.15}$$

1.1.6 Determination of control achieving regional controllability

The purpose of this section is to look for a control that achieves regional transfer with minimum energy. Obviously, we can use available results on the controllability of the distributed systems, but the difficulty appears is the desired state is given only on the region. Moreover, we have shown that the regional transfer cost is minimal.
The problem is to transfer, with minimal cost, the system (1.1) from y_0 to y_d at time T. For this, consider the following set

$$G = \{g \in \mathcal{V} \quad \text{s.t.} \quad g = 0 \ \text{ on } \omega\}. \tag{1.16}$$

Thus, the question of the regional transfer becomes
Is there a minimal energy control $u \in U$ such as

$$y_u(T) - y_d \in G?$$

To solve this problem, consider the following

$$U_{ad} = \{u \in U : y_u(T) - y_d \in G\}.$$

The problem of regional controllability with minimal energy can be formulated as follows

$$\inf_{u \in U_{ad}} \|u\|^2. \tag{1.17}$$

To solve this problem we propose the following approach.

General approuche

Let's consider the system (1.1) and consider the following

$$\hat{G} = \{g \in \mathcal{V}' \text{ s.t.} \quad g = 0 \ \text{ on } \Omega \backslash \omega\}. \tag{1.18}$$

For $\varphi_0 \in \hat{G}$, consider the following system in $\mathcal{V}'$

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}(t) = A^* \varphi(t) \ \ t \in]0, T[, \\ \varphi(T) = \varphi_0, \end{cases} \tag{1.19}$$

and the application

$$\|\varphi_0\|^2_{\hat{G}} = \int_0^T \|B^* \varphi(t)\|^2 \, dt. \tag{1.20}$$

Consider again the system

$$\begin{cases} \dfrac{\partial \psi}{\partial t}(t) = A\psi(t) + BB^{*}\varphi(t) \ \ t \in]0, T[\,, \\ \psi(0) = y_0. \end{cases} \tag{1.21}$$

We define the operator $\mathcal{M}$ as

$$\mathcal{M}\varphi_0 = \mathcal{P}(\psi(T)), \tag{1.22}$$

where $\mathcal{P} = \chi_{\omega}^{*}\chi$. The $\mathcal{M}$ operator is an affine operator that can be decomposed as follows

$$\mathcal{M}\varphi_0 = \mathcal{P}(\psi_0(T) + \psi_1(T)),$$

where ψ_0 and $\psi_1$sont system solutions (1.23) and (1.24) respectively with

$$\begin{cases} \dfrac{\partial \psi_0}{\partial t}(t) = A\psi_0(t) \ \ t \in]0, T[\,, \\ \psi_0(0) = y_0, \end{cases} \tag{1.23}$$

and

$$\begin{cases} \dfrac{\partial \psi_1}{\partial t}(t) = A\psi_1(t) + BB^{*}\varphi(t), \ t \in]0, T[\,, \\ \psi_1(0) = 0. \end{cases} \tag{1.24}$$

With these various systems, we will consider the operator that leads us from $\varphi_0$to $\psi_1(T)$ through the following steps

$$\varphi_0 \longrightarrow \boxed{\text{by resolution of (1.19)}}\ \varphi \longrightarrow \boxed{\text{by resolution of (1.24)}}\ \psi_1 \longrightarrow \psi_1(T).$$

Then, consider

$$\Lambda\varphi_0 = \mathcal{P}(\psi_1(T)). \tag{1.25}$$

The operator Λ is bounded and symetric. Then for all $\varphi_0, \hat{\varphi}_0 \in \hat{G}$, we have

$$\langle \Lambda\varphi_0, \hat{\varphi}_0 \rangle = \langle \psi_1(T), \hat{\varphi}_0 \rangle = \int_0^T B^{*}\varphi(t)\, B^{*}\hat{\varphi} dt.$$

So the problem of regional controllability returns to the solution of the equation

$$\Lambda\varphi_0 = \chi_{\omega}^{*} y_d - \mathcal{P}(\psi_0(T)). \tag{1.26}$$

Then we have the following result

Proposition 1.7 *[15] Let ω be a given non-empty region of Ω. If the system (1.1) is weakly regionally controllable on ω then equation (1.26) has a single solution $\varphi_0 \in \hat{G}$. The control*

$$u^{*} = B^{*}\varphi(t). \tag{1.27}$$

transfer the system (1.1) in G at time T, i.e.

$$y_{u^*}(T)|_\omega - y_d \in G,$$

moreover, this control is the solution to the problem (1.17), i.e. it minimizes the cost of regional transfer.

$$\mathcal{J}(u) = \int_0^T \|u(t)\|^2 \, dt. \tag{1.28}$$

Proof.
*First we prove that if the system (1.1) is weakly regionally controllable on ω then (1.20) defines a norm. For $\varphi_0 \in \hat{G}$, we have

$$\|\varphi_0\| = 0 \Rightarrow \int_0^T \|B^*\varphi(t)\|^2 \, dt = 0 \Leftrightarrow B^*\varphi(t) = 0 \Leftrightarrow B^*S^*(T-t)\varphi_0 = 0.$$

The system (1.1) is weakly regionally controllable, then we have

$$\overline{\operatorname{Im}\chi_\omega \mathcal{H}} = L^2(\omega) \Leftrightarrow \ker \mathcal{H}^*\chi_\omega^* = \{0\}.$$

Therefore, the weak regional controllability implies that if $B^*S^*(T-t)\varphi_0 = 0$ then $\varphi_0 = 0$.
*Consider the operator Λ, we get

$$\langle \Lambda\varphi_0, \varphi_0 \rangle_{\hat{G}^*,\hat{G}} = \langle \mathcal{P}(\psi_1(T)), \varphi_0 \rangle = \langle \psi_1(T), \varphi_0 \rangle,$$

and we have

$$\psi_1(T) = \int_0^T S(T-s)BB^*\varphi(s)\, ds,$$

hence

$$\begin{aligned}
\langle \Lambda\varphi_0, \varphi_0 \rangle_{\hat{G}^*\hat{G}} &= \left\langle \int_0^T S(T-s)BB^*\varphi(s)\, ds, \varphi_0 \right\rangle \\
&= \int_0^T \langle B^*\varphi(s)\, ds, B^*S^*(T-s)\varphi_0 \rangle \\
&= \int_0^T \|B^*\varphi(s)\|^2 \, ds = \|\varphi_0\|_{\hat{G}}^2.
\end{aligned}$$

Therefore, $\Lambda : \hat{G} \to \hat{G}^*$ is a bijection. Then, the equation (1.26) has a single solution φ_0. We put $u^*(t) = B^*\varphi(t)$ in G. We have $y_u(T)|_\omega = y_d$.
*For the optimality of u, consider u and v dans U_{ad}, then, $y(T,u) - y_d$, $y(T,v) - y_d \in G$, hence $(y(T,v) - y(T,u))|_\omega = 0$.
Then

$$\langle \varphi_0, (y(T,v) - y(T,u)) \rangle = 0,$$

which is equivalent to

$$\left\langle \int_0^T S(T-s)\, B(v(s)-u(s))\, ds, \varphi_0 \right\rangle = 0.$$

Consider

$$\int_0^T \langle v(s)-u(s), B^* S^* (T-s)\, \varphi_0 \rangle\, ds = 0,$$

which gives finally

$$\int_0^T \langle B^* \varphi(s), v(s)-u(s) \rangle\, ds = 0,$$

then

$$\begin{aligned} \mathcal{J}'(u^*)(v-u) &= 2\int_0^T \langle u^*(t), v(t)-u(t)\rangle\, dt \\ &= 2\int_0^T \langle B^*\varphi(t), v(t)-u(t)\rangle\, dt \\ &= 0, \end{aligned}$$

because

$$\begin{aligned} \mathcal{J}(u^* + t(u-v)) &= \int_0^T \|u^*(t) + t(u(t)-v(t))\|^2\, dt \\ &= \int_0^T \|u^*(t)\|^2 + t^2 \|(u(t)-v(t))\|^2\, dt + 2\langle u^*(t), u(t)-v(t)\rangle\, dt \\ &\Rightarrow \frac{\mathcal{J}(u^*+t(u-v)) - \mathcal{J}\ (u^*)}{t} = \int_0^T t\,\|(u(t)-v(t))\|^2 + 2\langle u^*(t), u(t)-v(t)\rangle\, dt \\ &\Rightarrow \mathcal{J}'(u^*)(u-v) = 2\int_0^T \langle u^*(t), (u(t)-v(t))\rangle\, dt, \end{aligned}$$

which establishes the optimality of control u^*. ■

1.2 Optimal control

Optimal control, also known as trajectory optimization, seeks to determine the input (control) to a dynamical system or system defined by PDE (distributed parameter system) that optimize a given performance functional (maximize profit, minimize cost, etc), while satisfying different kinds of constraints.

Our main objective in this section will be to recall some classical results about optimal control of distributed parameter systems which introduced by J. L. Lions [45].

1.2.1 Problem statement

Let $\mathcal{V}$ and U be real Hilbert spaces of states and controls resp. Consider the optimal control problem in infinite dimension which is written in the following abstract form

$$\inf_{u \in U_{ad}} \mathcal{J}(u), \tag{1.29}$$

with constraints

$$Ay(u) = f + Bu, \tag{1.30}$$

where A be a linear partial differential operator stationary or evolutionary makes an isomorphism on $\mathcal{V}'$ identified to $\mathcal{V}$, $B \in \mathcal{L}(U, \mathcal{V})$ the control operator, $U_{ad} \in U$ a non empty closed convex subset of admissible controls, f is a source function in $\mathcal{V}$.
Let $\mathcal{Z}$ be a Hilbert space of observations. Consider the following quadratic cost function

$$\mathcal{J}(u) = \|Cy(u) - y_d\|_Z^2 + \langle Nu, u \rangle_U, \forall u \in U_{ad}, \tag{1.31}$$

where $C \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$ be the observations operator, y_d is a fixed observation in $\mathcal{Z}$ and N is a symmetric definite positive operator bounded on U. Then, our optimal control problem consists in characterizing the control u which minimizes $\mathcal{J}$ on U_{ad} i.e.

$$\begin{cases} \text{Find } u \in U_{ad} \text{ such as} \\ \mathcal{J}(u, y(u)) = \inf \mathcal{J}(v, y(v)), \forall v \in U_{ad}, \\ Ay(v) = f + Bv. \end{cases} \tag{1.32}$$

A control-state pair $(u, y(u))$ is called optimal pair if it solves (1.32).

1.2.2 Optimal control characterization

Theorem 1.1 *[23] If the cost function $\mathcal{J}$ is Gateaux-differentiable function, then the following necessary and sufficient optimality condition holds*

$$(\mathcal{J}'(u), (v-u))_U = (Cy(u) - y_d,\ Cy(v-u))_Z + N(u, v-u)_U \geq 0,\ \forall v \in U_{ad}, \tag{1.33}$$

*which called **variational inequality**.*

Optimality condition

Let $p = p(u)$ be the adjoint state, defined by

$$A^* p = C^*(Cy(u) - y_d), \tag{1.34}$$

where $C^* \in \mathcal{L}(\mathcal{Z}, \mathcal{V})$ the adjoint of C and A^* is the adjoint operator of A, then

$$\begin{aligned}(Cy(u) - y_d, Cy(v-u))_{\mathcal{Z}} &= (C^*(Cy(u) - y_d), y(v-u))_{\mathcal{V}}, \\ &= (A^*p, y(v+u))_{\mathcal{V}}, \\ &= (p, Ay(v-u))_{\mathcal{V}}, \\ &= (p, B(v-u))_{\mathcal{V}}, \\ &= (B^*p, v-u)_U.\end{aligned}$$

Hence variational inequality 1.33 is equivalent to

$$(B^*p + Nu, v-u)_U \geq 0, \quad \forall v \in U_{ad}, \tag{1.35}$$

and the unique optimal control u is given by the resolution of the following optimality system

$$\begin{cases} Ay(u) = f + Bu, \\ A^*p = C^*(Cy(u) - y_d), \\ \forall u \in U_{ad}, \\ (B^*p + Nu, v-u)_U \geq 0, \quad \forall v \in U_{ad}. \end{cases} \tag{1.36}$$

1.2.3 Application (Optimal control of hyperbolic systems)

Let Ω a bounded domain of $\mathbb{R}^n$ with boundary Γ of class C^2, $T > 0$, $Q = \Omega \times [0,T]$, $\Sigma = \Gamma \times [0,T]$. Let $U = L^2(Q)$ be the space of controls, U_{ad} is the set admissible controls non-empty closed and convex, B is abounded operator from U to $\mathcal{V} = L^2(0,T; H_0^1(\Omega))$. Consider the following second order hyperbolic PDE

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} + A(x)y = f + Bu & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x), \ \dfrac{\partial y}{\partial t}(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{1.37}$$

where $f \in L^2(Q)$, $y_0 \in H_0^1(\Omega), y_1 \in L^2(\Omega)$.
Our optimal control problem consists in looking for a control function $u \in U_{ad}$ which minimizes the following cost function

$$\mathcal{J}(v) = \|Cy(v) - y_d\|_{\mathcal{Z}}^2 + (Nv, v)_U, \ \forall v \in U_{ad}, \tag{1.38}$$

i.e.

$$\inf_{v \in U_{ad}} \mathcal{J}(v, y(v)). \tag{1.39}$$

In this case, consider the observation space $\mathcal{Z} = L^2(Q)$.The observation operator is

$$\mathcal{C} : L^2(0,T;H_0^1(\Omega)) \longrightarrow L^2(Q).$$

Theorem 1.2 *[45] The optimal control u solution of (1.37), (1.39) is unique and it's characterized by the following optimality system*

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} + A(x)y = f + Bu & \textit{in } Q, \\ y = 0 & \textit{on } \Sigma, \\ y(x,0) = y_0(x), \dfrac{\partial y}{\partial t}(x,0) = y_1(x) & \textit{in } \Omega, \\ \dfrac{\partial^2 p}{\partial t^2} + A(x)p = C^*\left(Cy(u) - y_d\right) & \textit{in } Q, \\ p = 0 & \textit{on } \Sigma, \\ p(x,T) = 0, \dfrac{\partial p}{\partial t}(x,T) = 0 & \textit{in } \Omega, \end{cases} \tag{1.40}$$

with the variational inequality

$$(B^*p + Nu, v - u)_U \geq 0, \ \forall v \in U_{ad}. \tag{1.41}$$

Proof. An optimality condition is written as follow:

$$\mathcal{J}'(u)(v-u) = (Cy(u) - y_d, \ Cy(v) - Cy(u))_{\mathcal{Z}} + N(u, v-u)_U \geq 0, \ \forall v \in U_{ad}. \tag{1.42}$$

Introduce the adjoint state given by

$$\begin{cases} \dfrac{d^2 p}{dt^2} + A^*p = C^*\left(Cy(u) - y_d\right) & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p\,(x,T) = 0, \ \dfrac{dp}{dt}\,(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{1.43}$$

Now, let's rewrite the first order Euler (1.42) condition as

$$\begin{aligned} \left(C^*\left(Cy(u) - y_d\right), \ y(v) - y(u)\right)_{\mathcal{V}} &= \int_0^T \int_\Omega \left\langle \frac{d^2 p}{dt^2} + A^*p, y(v) - y(u) \right\rangle dxdt \\ &= \int_0^T \int_\Omega (p, B(v-u))\, dxdt, \end{aligned}$$

so, we get (1.41). ■

1.3 Averaged controllability

In many physical processes, modeling by parameter dependent system appeared to be a challenging problem. In such a situation where the parameter has an unknown value, we cannot control every realization of the system by a single control (using a control independent of the parameter). In this context, Zuazua in [69] introduced the notion of averaged controllability. Its goal is to control the averaged state of a parameterized system instead of the state with respect to the unknown parameter.

In this section, we present the notion of averaged controllability for the infinite dimensional systems and we give its characterizations and properties, after that we study the optimal control problem for parameter dependent hyperbolic systems and we characterize the averaged optimal control.

1.3.1 Problem statement

Let Ω be an open subset of $\mathbb{R}^n$ with a regular boundary Γ and consider a time interval $]0,T[$, whith $T>0$. We denote by $Q=\Omega\times]0,T[$, $\Sigma=\Gamma\times]0,T[$ and we consider the following control system depending on an unknown parameter σ

$$\begin{cases} \dfrac{dy}{dt}=A(\sigma)y+B(\sigma)u, & \text{in } Q,\\ y(0)=y_0, & \text{in } \Omega, \end{cases} \tag{1.44}$$

where the operator $A(\sigma)$ depends on the uncertainty parameter σ and generates a strongly continuous semi-group $\{S(t,\sigma)\}_{t\geq 0}$ on the Hilbert state space $\mathcal{V}$, $B\in L(U,\mathcal{V})$, $u=u(t,x)\in U$ is a distributed control which doesn't depend on σ, U is space of controls, the initial data $y_0(x)\in\mathcal{V}$ is independent of the parameter σ. To simplify the notation we will assume that $\sigma\in(0,1)$.

The solution of the differential system (1.44) can be represented as follows

$$y(t,\sigma)=S(t,\sigma)\,y_0+\int_0^t S(t-s,\sigma)\,B(\sigma)\,u(s)\,ds. \tag{1.45}$$

1.3.2 Exact, weak & null averaged controllability

Definition 1.6 *The system (1.44) is said to be exactly averaged controllable in time $T>0$ if, for every final target $y_d\in\mathcal{V}$, there exists a control $u\in L^2(0,T;U)$ independent of the parameter σ such that*

$$\int_0^1 y(T,\sigma)\,d\sigma=y_d. \tag{1.46}$$

*This notion is illustrated on **Figure 4**.*

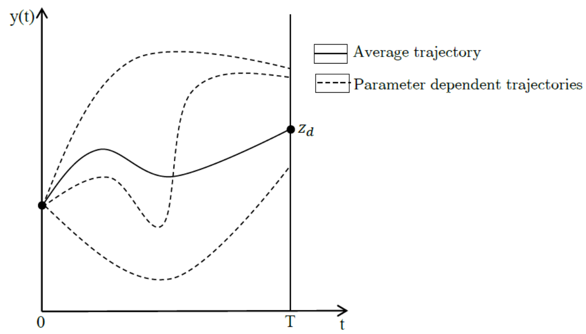

***Figure 4**: Exact averaged controllability.*

Definition 1.7 *The system (1.45) is said to be weakly averaged controllable in time $T > 0$ if, for every $\varepsilon > 0$ and every final target $y_d \in \mathcal{V}$, there exists a control $u \in L^2(0, T; U)$ such that*

$$\left\| \int_0^1 y(T, \sigma)\, d\sigma - y_d \right\|_{\mathcal{V}} < \varepsilon. \tag{1.47}$$

*This notion is illustrated on **Figure 5**.*

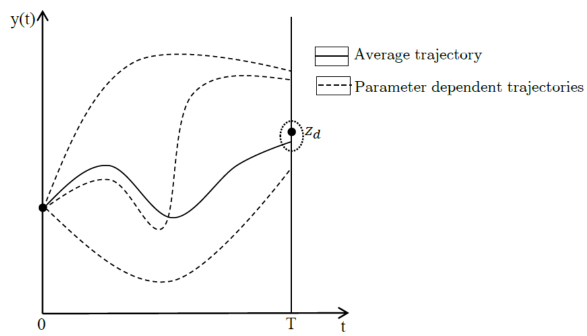

***Figure 5**: Weak averaged controllability.*

Definition 1.8 *The system (1.44) is said to be null averaged controllable in time $T > 0$ if there exists a control $u \in L^2(0, T; U)$ independent of the parameter σ such that*

$$\int_0^1 y(T, \sigma)\, d\sigma = 0. \tag{1.48}$$

*This notion is illustrated on **Figure 6**.*

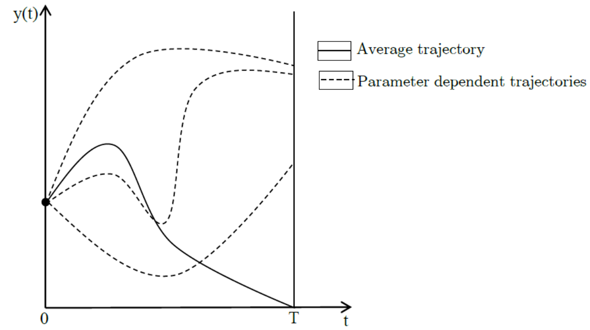

Figure 6: Null averaged controllability.

Remark 1.4 *This concept of averaged controllability differs from that of simultaneous controllability in which one is interested on controlling all states simultaneously and not only its average.*

1.3.3 The averaged controllability operator

For the system (1.44), consider $\mathcal{H}_A : L^2([0,T];U) \rightarrow \mathcal{V}$ the operator defined by

$$\mathcal{H}_A u = \int_0^1 \int_0^T S\left(T-t,\sigma\right) B\left(\sigma\right) u\left(t\right) dt d\sigma, \tag{1.49}$$

and its adjoint $\mathcal{H}_A^* : \mathcal{V} \rightarrow L^2([0,T];U)$ is given by

$$\mathcal{H}_A^* = \int_0^1 S^*\left(T-.,\sigma\right) B^*\left(\sigma\right) d\sigma. \tag{1.50}$$

Where B^* (resp. $S^*\left(T-.,\sigma\right)$) is the adjoint of of B (resp. $S\left(T-.,\sigma\right)$).
$\mathcal{H}_A$ will be used later to obtain various definitions and properties of averaged controllability. For the study of controllability, without loss generality, we can assume that $y_0 = 0$.

Proposition 1.8
i) *The system (1.44) is exactly averaged controllable iff $\mathcal{H}_A$ is surjective i.e.* $\operatorname{Im}\mathcal{H}_A = \mathcal{V}$.
ii) *The system (1.44) is weakly averaged controllable iff the image of $\mathcal{H}_A$ is dense i.e.* $\overline{\operatorname{Im}\mathcal{H}_A} = \mathcal{V}$.
Proof.
i) *The system (1.44) is exactly averaged controllable*

$$\begin{aligned}
&\Longleftrightarrow \quad \forall y_0, y_d \in \mathcal{V}, \exists u \in L^2(0,T;U) \ \text{s.t.}\ y_d = \int_0^1 y\left(T,\sigma\right) d\sigma = \int_0^1 \int_0^T S\left(T-t,\sigma\right) B\left(\sigma\right) u\left(t\right) dt d\sigma,\\
&\Longleftrightarrow \quad \mathcal{H}_A \text{ is surjectif},\\
&\Longleftrightarrow \quad \operatorname{Im}\mathcal{H}_A = \mathcal{V}.
\end{aligned}$$

ii) *The system (1.44) is weakly averaged controllable*

$$\Longleftrightarrow \quad \forall y_0, y_d \in \mathcal{V}, \exists u \in L^2(0,T;U) : \left\| \int_0^1 y\left(T,\sigma\right) d\sigma - y_d \right\|_{\mathcal{V}} < \varepsilon, \forall \varepsilon > 0,$$

$$\Longleftrightarrow \quad \forall y_d \in \mathcal{V}, \exists u \in L^2(0,T;U) : \left\| \mathcal{H}_A u - (\int_0^1 S\left(T,\sigma\right) y_0 d\sigma + y_d) \right\|_{\mathcal{V}} < \varepsilon, \forall \varepsilon > 0,$$

$$\Longleftrightarrow \quad \operatorname{Im} \mathcal{H}_A \textit{ is dense in } \mathcal{V}.$$

■

Proposition 1.9 *The system (1.44) is exactly averaged controllable in a time T iff*

$$\exists \gamma > 0, \forall y \in \mathcal{V} : \int_0^T \left\| \int_0^1 B^*(\sigma) S^*\left(T-t,\sigma\right) y d\sigma \right\|_U^2 dt \geq \gamma \left\| y \right\|_{\mathcal{V}}^2. \tag{1.51}$$

*This last inequality is called "**averaged observability inequality**".*

Proof. According to the proposition (1.8) the exact averaged controllability equivalent to $\operatorname{Im} \mathcal{H}_A = \mathcal{V}$.

$\mathcal{H}_T^* : \mathcal{V} \longrightarrow L^2\left(0,T;U\right)$ is continuously invertible (see Theorem 3 in Appendix) i.e

$$\exists \gamma > 0, \forall y \in \mathcal{V} : \int_0^T \left\| \mathcal{H}_A^* y \right\|_U^2 dt > \gamma \left\| y \right\|_{\mathcal{V}}^2.$$

■

We introduce the following matrix

$$\mathcal{G}_A = \mathcal{H}_A \mathcal{H}_A^* = \int_0^T \int_0^1 S\left(T-t,\sigma\right) B\left(\sigma\right) d\sigma \int_0^1 S^*\left(T-t,\sigma\right) B^*\left(\sigma\right) d\sigma dt, \tag{1.52}$$

which is the **average controllability Gramian**.

1.3.4 The explicit formula of control achieving averaged controllability

As $\{S(t,\sigma)\}_{t\geq 0}$ and $B\left(\sigma\right)$ are assumed to be continuous for all $\sigma \in (0,1)$, then the average controllability Gramian verify

$$\exists c > 0 \text{ s.t. } \left\| \mathcal{G}_A y \right\|_{\mathcal{V}}^2 \leq \int_0^T \left\| \int_0^1 S(T-t,\sigma) B(\sigma) d\sigma \int_0^1 B^*(\sigma) S^*(T-t,\sigma) y d\sigma \right\|_{\mathcal{V}}^2 dt \ \leq c \left\| y \right\|_{\mathcal{V}}^2, \forall y \in \mathcal{V}. \tag{1.53}$$

It is also self-adjoint and no negative since

$$\langle \mathcal{G}_A y, y \rangle_{\mathcal{V}} = \int_0^T \left\| \int_0^1 B^*(\sigma) S^*(T-t,\sigma) y d\sigma \right\|_{\mathcal{V}}^2 dt \geq 0 \ , \forall y \in \mathcal{V}. \tag{1.54}$$

which implies the existence of a self-adjoin and no negative operator $\sqrt{\mathcal{G}_A}$ also whose square equal to $\mathcal{G}_A$.

Theorem 1.3
i) *There exists a control* $u \in L^2(0,T;U)$ *which steer* y_0 *to* y_d *in time* T *iff*

$$\int_0^1 S(T,\sigma)y_0 d\sigma - y_d \in \operatorname{Im}\sqrt{\mathcal{G}_A}. \tag{1.55}$$

ii) *Among the controls which transfer* y_0 *to* y_d *in time* T*, there exists a single control* u *which minimizes the functional*

$$\mathcal{J}(u) = \int_0^T \|u(t)\|_U^2 \, dt. \tag{1.56}$$

iii) *If* $\int_0^1 S(T,\sigma)y_0 d\sigma - y_d \in \operatorname{Im}\sqrt{\mathcal{G}_A}$*, then* u *is given by*

$$\hat{u} = -\int_0^1 B^*(\sigma)S^*(T-t,\sigma)\mathcal{G}_T^{-1}\left(\int_0^1 S(T,\sigma)y_0 d\sigma - y_d\right) d\sigma, \forall t \in [0,T]. \tag{1.57}$$

1.3.5 Application (Averaged control of parameter dependent hyperbolic systems)

Consider the following abstract hyperbolic problem

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} + A(\sigma)y = f + B(\sigma)u & \text{in } Q, \\ y(x,0) = y_0, \dfrac{\partial y}{\partial t}(x,0) = y_1 & \text{in } \Omega, \end{cases} \tag{1.58}$$

Let $\mathcal{Z}, \mathcal{V}$ be Hilbert spaces with $\mathcal{Z}$ is separable and dense in $\mathcal{V}$, $f \in L^2(0,T;\mathcal{V})$, $y_0 \in \mathcal{Z}$, $y_1 \in \mathcal{V}$, $B(\sigma) \in \mathcal{L}(U_{ad}, L^2(0,T;\mathcal{V}))$ and $U_{ad} \subset L^2(0,T;\mathcal{V})$. Then, for every $\sigma \in (0,1)$ the equation (1.58) has aunique solution in $L^2(0,T;\mathcal{V})$ [36].

Let $\int_0^1 y(x,t,\sigma)d\sigma \in L^2(0,T;\mathcal{Z})$ be the averaged state respect to σ and y_d a given desired state in $L^2(0,T;\mathcal{Z})$ too, we are interested to the following optimal control problem

$$\inf_{v \in U_{ad}} \mathcal{J}(v) \text{ with } \mathcal{J}(v) = \left\| \int_0^1 y(x,t,\sigma)d\sigma - y_d \right\|^2_{L^2(0,T;\mathcal{Z})} + N\|v\|^2_{L^2(0,T;\mathcal{Z})}, \tag{1.59}$$

with $N > 0$.

Theorem 1.4 *The averaged optimal control* u *solution of (1.58) (1.59) is unique and it's characterized by*

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} + A(x,\sigma)y = f + B(\sigma)u & \text{in } Q, \\ \dfrac{\partial^2 p}{\partial t^2} + A^*(x,\sigma)p = \int_0^1 y(u,t,\sigma)d\sigma - y_d & \text{in } Q, \\ y(x,0) = y_0(x), \dfrac{\partial y}{\partial t}(x,0) = y_1(x), p(x,T) = 0, \dfrac{\partial p}{\partial t}(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{1.60}$$

with the variational inequality

$$\int_0^T \int_\Omega \int_0^1 \left(B^*(\sigma)\,p + Nu, v-u\right) d\sigma dx dt \geq 0,\ \forall v \in U_{ad}. \tag{1.61}$$

Proof. An optimality condition is written as follow

$$\begin{aligned} \mathcal{J}'(u)(v-u) &= \left(\int_0^1 y(x,t,u,\sigma)d\sigma - y_d,\ \int_0^1 y(x,t,v,\sigma)d\sigma - \int_0^1 y(x,t,u,\sigma)d\sigma\right)_{\mathcal{V}} \\ +N(u,v-u)_U &\geq 0,\ \forall v \in U_{ad}. \end{aligned} \tag{1.62}$$

Introduce the following adjoint state

$$\begin{cases} \dfrac{\partial^2 p}{\partial t^2} + A^*(x,\sigma)\,p = \displaystyle\int_0^1 y(x,t,\sigma)d\sigma - y_d & \text{in } Q, \\ \varphi(x,T) = 0,\ \dfrac{\partial p}{\partial t}(x,T) = 0 & \text{in } \Omega. \end{cases}$$

Now, let's rewrite the first order Euler (1.62) condition as

$$\begin{aligned} &\int_0^T \int_\Omega \left(\int_0^1 y(x,t,u,\sigma)d\sigma - y_d, \int_0^1 y(x,t,v,\sigma)d\sigma - \int_0^1 y(x,t,u,\sigma)d\sigma\right) dxdt \\ &= \int_0^T \int_\Omega \int_0^1 \left(\frac{d^2p}{dt^2} + A^*(x,\sigma)\,p, y(v) - y(u)\right) d\sigma dx dt \\ &= \int_0^T \int_\Omega \int_0^1 \left(p, B(\sigma)(v-u)\right) d\sigma dx dt. \end{aligned}$$

Hence, we get (1.61). ■

1.4 No-regrets control & low-regret control for stationary problems

The concepts of no-regret control and low regret control were developed by Lions [40], [44] for controlling systems modeled by partial differential equations with missing data or with partial information.

In what follows, we are devoted to the definition of the concept of Pareto control, no regrets control, and low regrets control for a stationary problem, followed by application on a case of hyperbolic systems.

1.4.1 Problem statement

Let G is a non-empty closed subspace of Hilbert space of uncertainties F and let $\beta \in \mathcal{L}(F, \mathcal{V}')$. For $f \in \mathcal{V}'$, the equation for the control $v \in U$ and the uncertainty $g \in G$ is given by

$$Ay(v, g) = f + Bv + \beta g, \tag{1.63}$$

where the theoretical basis remains the same as the one given in the section above.
The problem (1.63) is well posed in $\mathcal{V}$ and therefore it has a single solution denoted $y = y(v, g)$. For every $g \in G$, we have a possible state to which we attach the following cost function

$$\mathcal{J}(v, g) = \|Cy(v, g) - y_d\|^2_{\mathcal{Z}} + N \|v\|^2_U, \tag{1.64}$$

where $C \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$, $\mathcal{Z}$ is the observation space, $y_d \in \mathcal{Z}$ and $N > 0$.
We are concerned with the optimal control of the problem (1.63) (1.64), i.e. we want to solve the problem

$$\inf_{v \in U} \mathcal{J}(v, g). \tag{1.65}$$

When $G = \{0\}$, then the problem (1.65) becomes a standard optimal control problem.
When $G \neq \{0\}$, then the problem (1.65) has no sense when G is an infinite dimensional space.
The classical idea is to proceed to the calculation of

$$\inf_{v \in U} \left(\sup_{g \in G} \mathcal{J}(v, g) \right). \tag{1.66}$$

but $\mathcal{J}(v, g)$ hasn't anupper bound because $\sup\limits_{g \in G} \mathcal{J}(v, g) = +\infty$.
To avoid difficulty arises in (1.66) Lions thought to look only for controls v such that $\mathcal{J}(v, g) \leq \mathcal{J}(0, g)$,$\forall g \in G$.

1.4.2 Pareto control

Definition 1.9 *[55] We say that $u \in U$ is a Pareto control for the system (1.63) (1.64) iff*

$$\begin{cases} \mathcal{J}(u, g) \leq \mathcal{J}(v, g) \ \forall v \in U, \forall g \in G, \\ \textit{and if there exists } g_0 \in G \textit{ s.t.} \\ \mathcal{J}(u, g_0) < \mathcal{J}(v, g_0) \ \forall v \in U. \end{cases} \tag{1.67}$$

Definition 1.10 *[55] We say that a Pareto control $u \in U$ is related to $u_0 \in U$ if*

$$\mathcal{J}(u, g) \leq \mathcal{J}(u_0, g) \ \forall g \in G. \tag{1.68}$$

1.4.3 No-regret control

Definition 1.11 *We say that $u \in U$ is a no-regrets control related to u_0 for (1.63) (1.64) if u is the solution to the following problem*

$$\inf_{v\in U}\left(\sup_{g\in G}\left(\mathcal{J}(v,g)-\mathcal{J}(u_0,g)\right)\right). \tag{1.69}$$

Remark 1.5 *When $u_0 = 0$, the definition coincides with that of the no-regrets control defined by Lions [40]*

Remark 1.6 *Obviously, the problem (1.69) is defined only for the controls $v \in U$ such that*

$$\sup_{g\in G}\left(\mathcal{J}(v,g)-\mathcal{J}(u_0,g)\right)<\infty. \tag{1.70}$$

Lemma 1.1 *[55] For any u_0 fixed in U and for any $v \in U$ we have*

$$\mathcal{J}(v,g)-\mathcal{J}(u_0,g)=\mathcal{J}(v,0)-\mathcal{J}(u_0,0)+2\left\langle \beta^*\zeta\left(v-u_0\right),g\right\rangle_{G',G}, \tag{1.71}$$

where $\zeta(v) \in \mathcal{V}$ is defined for $v \in U$ by

$$A^*\zeta(v)=C^*C\left(y(v,0)-y(0,0)\right). \tag{1.72}$$

Proof. Due A is an isomorphism, we have

$$y(v,g)=y(v,0)+y(0,g)-y(0,0). \tag{1.73}$$

From the definition of $\mathcal{J}$ and (1.73), we have

$$\begin{aligned}\mathcal{J}(v,g) &= \left\|C\left(y(v,0)+y(0,g)-y(0,0)\right)-y_d\right\|_{\mathcal{Z}}^2+N\left\|v\right\|_U^2\\ &= \mathcal{J}(v,0)+\left\|C\left(y(0,g)-y(0,0)\right)\right\|_{\mathcal{Z}}^2+2(Cy(v,0)-y_d,C\left(y(0,g)-y(0,0)\right)_{\mathcal{Z}},\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}(u_0,g) &= \left\|C\left(y(u_0,0)+y(0,g)-y(0,0)\right)-y_d\right\|_{\mathcal{Z}}^2+N\left\|v\right\|_U^2\\ &= \mathcal{J}(u_0,0)+\left\|C\left(y(0,g)-y(0,0)\right)\right\|_{\mathcal{Z}}^2+2(Cy(u_0,0)-y_d,C\left(y(0,g)-y(0,0)\right))_{\mathcal{Z}}\end{aligned}$$

it comes

$$\begin{aligned}\mathcal{J}(v,g)-\mathcal{J}(u_0,g) &= \mathcal{J}(v,0)-\mathcal{J}(u_0,0)+2(C(y(v-u_0,0)-y(0,0)),C(y(0,g)-y(0,0)))_{\mathcal{Z}}\\ &= \mathcal{J}(v,0)-\mathcal{J}(u_0,0)+2(C^*C(y(v-u_0,0)-y(0,0)),y(0,g)-y(0,0))_{\mathcal{V}}.\end{aligned}$$

Introduce the state $\zeta(v)$ given by (1.72) to write

$$\begin{aligned}\mathcal{J}(v,g)-\mathcal{J}(u_0,g) &= \mathcal{J}(v,0)-\mathcal{J}(u_0,0)+2\left(A^*\zeta\left(v-u_0\right),y(0,g)-y(0,0)\right)_{\mathcal{V}}\\ &= \mathcal{J}(v,0)-\mathcal{J}(u_0,0)+2\left(\zeta\left(v-u_0\right),\beta g\right)_{\mathcal{V}}\\ &= \mathcal{J}(v,0)-\mathcal{J}(u_0,0)+2\left\langle\beta^*\zeta\left(v-u_0\right),g\right\rangle_{G',G}.\end{aligned}$$

Therefore the equation (1.71) is verified. ■

Remark 1.7 *To simplify, we define the operator S such that $S(v)=\beta^*\zeta(v)$ for $v\in U$. Then, we have*

$$\mathcal{J}(v,g)-\mathcal{J}(u_0,g)=\mathcal{J}(v,0)-\mathcal{J}(u_0,0)+2\left\langle S\left(v-u_0\right),g\right\rangle_{G',G}\ \forall g\in G. \tag{1.74}$$

Remark 1.8 *By (1.74) it is easy to verify that condition (1.70) holds for the no-regret control v iff $v\in K+u_0$, where*

$$K=\left\{w\in U:\langle S(w),g\rangle_{G',G}=0\ \forall g\in G\right\}.$$

Proposition 1.10 *[55] Let $u_0\in U$. Then there is a unique Pareto control related to u_0. Moreover, it is the unique element of the set $K+u_0$, which minimizes the functional $\mathcal{J}(v,0)$ on $K+u_0$.*

Theorem 1.5 *Let $u_0\in U$ be a given control, then we have $u\in U$ is a Pareto control related to u_0 iff u is a no-regrets control related to u_0.*

Proof. Let u be a Pareto control related to u_0, and let $v\in K+u_0$. Then

$$\langle S(u-u_0),g\rangle_{G',G}=0=\langle S(v-u_0),g\rangle_{G',G}\ \forall g\in G,$$

and we have $\mathcal{J}(u,0)\leq\mathcal{J}(v,0)$ according to proposition. Thus, using (1.74)

$$\mathcal{J}(u,0)-\mathcal{J}(u_0,0)+2\left\langle S\left(u-u_0\right),g\right\rangle_{G',G}\leq\sup_{g\in G}\left(\mathcal{J}(v,g)-\mathcal{J}(u_0,g)\right)\ \forall g\in G.$$

Hence

$$\sup_{g\in G}\left(\mathcal{J}(u,g)-\mathcal{J}(u_0,g)\right)\leq\sup_{g\in G}\left(\mathcal{J}(v,g)-\mathcal{J}(u_0,g)\right)\ \forall g\in G.$$

Then,

$$\sup_{g\in G}\left(\mathcal{J}(u,g)-\mathcal{J}(u_0,g)\right)=\inf_{v\in K+u_0}\left(\sup_{g\in G}\left(\mathcal{J}(v,g)-\mathcal{J}(u_0,g)\right)\right)\ \forall g\in G.$$

Now, let $v\in U_{ad}\backslash K+u_0$. There exists $g_0\in G$ such that $\langle S(v-u_0),g\rangle_{G',G}\neq 0$. Then, we have

$$\sup_{g\in G}\left(\mathcal{J}(v,g)-\mathcal{J}(u_0,g)\right)=\mathcal{J}(v,0)-\mathcal{J}(u_0,0)+2\sup_{g\in G}\left\langle S\left(v-u_0\right),g\right\rangle_{G',G}=+\infty.$$

Note that G is a vector space, and we have only two possibilities:

$$\sup_{g\in G}\left\langle S\left(w\right),g\right\rangle_{G',G}=0 \text{ or } \sup_{g\in G}\left\langle S\left(w\right),g\right\rangle_{G',G}=+\infty.$$

In this case, $\lim\limits_{t\to\infty}\left\langle S\left(w\right),tg\right\rangle_{G',G}=0$.
On the other hand, since u is a Pareto control, we have $\mathcal{J}(u,g)-\mathcal{J}(u_0,g)\leq 0\ \forall g\in G$, hance

$$\mathcal{J}(u,g)-\mathcal{J}(u_0,g)\leq 0\leq \sup_{g\in G}\left(\mathcal{J}(v,g)-\mathcal{J}(u_0,g)\right)\ \forall g\in G.$$

Finally,

$$\sup_{g\in G}\left(\mathcal{J}(u,g)-\mathcal{J}(u_0,g)\right)\leq \inf_{v\in K+u_0}\left(\sup_{g\in G}\left(\mathcal{J}(v,g)-\mathcal{J}(u_0,g)\right)\right)\ \forall g\in G.$$

In conclusion, u is a no-regrets control related to u_0.
Conversely: Let u a no-regrets control related to u_0. We have

$$\sup_{g\in G}\left(\mathcal{J}(u,g)-\mathcal{J}(u_0,g)\right)\leq \sup_{g\in G}\left(\mathcal{J}(v,g)-\mathcal{J}(u_0,g)\right)\ \forall v\in U.$$

Then for $v=u_0$ and take (1.74) into account to find

$$\mathcal{J}(u,0)+2\sup_{g\in G}\left\langle S\left(u-u_0\right),g\right\rangle_{G',G}\leq \mathcal{J}(u_0,0)=c\ \forall g\in G,$$

where c is a constant.
We have $\mathcal{J}(u,0)\geq 0$ then, $\sup\limits_{g\in G}\left\langle S\left(u-u_0\right),g\right\rangle_{G',G}\leq c$. We deduce that $\sup\limits_{g\in G}\left\langle S\left(u-u_0\right),g\right\rangle_{G',G}=0$.
Therefore, $\sup\limits_{g\in G}\left\langle S\left(u-u_0\right),g\right\rangle_{G',G}\leq 0\ \forall g\in G$,then $\left\langle S\left(u-u_0\right),g\right\rangle_{G',G}\leq 0$, Thus $u\in K+u_0$ and we have

$$\mathcal{J}(u,0)\leq \mathcal{J}(v,0)\ \forall u\in K+u_0.$$

Finally, we conclude that u is a Pareto control related to u_0. ■

1.4.4 Low-regret control

Definition 1.12 *[44] We say that $u_\gamma\in U$ is a low-regret control related to u_0 for (1.63) (1.64) if u_γ is the solution to the following problem*

$$\inf_{v\in U}\sup_{g\in G}(\mathcal{J}(v,g)-\mathcal{J}(u_0,g)-\gamma\left\|g\right\|_G^2),\ \ \gamma>0. \tag{1.75}$$

The problem (1.69) is relaxed by introducing for $\gamma>0$, see [55] and [57] , the problem

$$\mathcal{J}(v,g)\leq \mathcal{J}(u_0,g)+\gamma\left\|g\right\|_G^2,\ \ \gamma>0,$$

hence

$$\mathcal{J}(v,g)-\mathcal{J}(u_0,g)-\gamma\left\|g\right\|_G^2=\mathcal{J}(v,0)-\mathcal{J}(u_0,0)+2\left\langle S(v-u_0),g\right\rangle_G-\gamma\left\|g\right\|_G^2,$$

which implies

$$\sup_{g\in G}(\mathcal{J}(v,g)-\mathcal{J}(u_0,g)-\gamma\left\|g\right\|_G^2)=\mathcal{J}(v,0)-\mathcal{J}(u_0,0)+\sup_{g\in G}(2\left\langle S(v-u_0),g\right\rangle_G-\gamma\left\|g\right\|_G^2),$$

by using Legendre transform [6] we obtain

$$\sup_{g\in G}(\mathcal{J}(v,g)-\mathcal{J}(u_0,g)-\gamma\left\|g\right\|_G^2)=\mathcal{J}(v,0)-\mathcal{J}(u_0,0)+\frac{1}{\gamma}\left\|S(v-u_0)\right\|_G^2.$$

Then,

$$\inf_{v\in U}\mathcal{J}^\gamma(v), \tag{1.76}$$

where

$$\mathcal{J}^\gamma(v)=\mathcal{J}(v,0)-\mathcal{J}(u_0,0)+\frac{1}{\gamma}\left\|S(v-u_0)\right\|_G^2. \tag{1.77}$$

Existence and uniqueness of low-regret control

Theorem 1.6 *Problem (1.76) (1.77) has a single solution u_γ called low-regret control related to u_0*

Proof. From the definition of $\mathcal{J}^\gamma$,we have that

$$\mathcal{J}^\gamma(v)\geq-\mathcal{J}(u_0,0)\,\forall v\in U.$$

i.e, $d_\gamma=\inf_{v\in U}\mathcal{J}^\gamma(v)$ exists.
Let a minimizing sequence (v_n^γ) (See Appendix) such that $d_\gamma=\lim_{i\to\infty}J^\gamma(v_n^\gamma)$, we have

$$-\mathcal{J}(u_0,0)\leq\mathcal{J}^\gamma(v_n^\gamma)=\mathcal{J}(v_n^\gamma,0)-\mathcal{J}(u_0,0)+\frac{1}{\gamma}\left\|S(v_n^\gamma-u_0)\right\|_G^2\leq d_\gamma+1,$$

which implies that

$$\left\|Cy(v_n^\gamma,0)-y_d\right\|_{\mathcal{Z}}^2+N\left\|v_n^\gamma\right\|_U^2+\frac{1}{\gamma}\left\|S(v_n^\gamma-u_0)\right\|_G^2\leq d_\gamma+\mathcal{J}(0,u_0)+1=C_\gamma.$$

we deduce that

$$\begin{aligned}\left\|v_n^\gamma\right\|_U&\leq C_\gamma,\\ \left\|Cy(v_n^\gamma,0)-y_d\right\|_{\mathcal{Z}}&\leq C_\gamma,\text{ implies }\left\|Cy(v_n^\gamma,0)\right\|_{\mathcal{Z}}\leq C_\gamma,\\ \left\|S(v_n^\gamma-u_0)\right\|_G&\leq C_\gamma\sqrt{\gamma},\end{aligned} \tag{1.78}$$

where the constant C_γ (independent of n) is not always the same.
From (1.78) we deduce that (v_n^γ) is bounded in compact space U, then we can extracting a subsequence still denoting by (v_n^γ) converges weakly to u_γ in U, due to isomorphism of A we deduce that $y\left(v_n^\gamma, 0\right)$ converge weakly to $y\left(u_\gamma, 0\right)$ in $\mathcal{V}$.
The cost function $\mathcal{J}^\gamma\left(v\right)$ is a lower semi continuous

$$\mathcal{J}^\gamma\left(u_\gamma\right) \leq \lim_{n\to\infty} \inf_{v\in U} \mathcal{J}^\gamma\left(v_n^\gamma\right) = \inf_{v\in U} \mathcal{J}^\gamma\left(v\right) = d_\gamma,$$

and we deduce from the convexity strict of the cost function $\mathcal{J}$ that u_γ is unique. ■

Theorem 1.7 *The solution u_γ of the relaxed problem (1.76) (1.77)(The unique low-regret control) converges weakly in U to the no-regrets control related to u_0 when γ tends to 0.*

Proof. Let u_γ be the solution u_γ of the problem (1.76) (1.77) in U then for all $v \in U$

$$\mathcal{J}\left(u_\gamma, 0\right) - \mathcal{J}\left(u_0, 0\right) + \frac{1}{\gamma}\left\|S(u_\gamma - u_0)\right\|_G^2 \leq \mathcal{J}\left(v, 0\right) - \mathcal{J}\left(u_0, 0\right) + \frac{1}{\gamma}\left\|S(v - u_0)\right\|_G^2 \quad \forall v \in U,$$

In particular for $v = u_0$ we have

$$\mathcal{J}\left(u_\gamma, 0\right) - \mathcal{J}\left(u_0, 0\right) + \frac{1}{\gamma}\left\|S(u_\gamma - u_0)\right\|_G^2 \leq 0,$$

and the structure of $J\left(u_\gamma, 0\right)$ in (1.64) gives

$$\left\|Cy\left(u_\gamma, 0\right) - y_d\right\|_{\mathcal{Z}}^2 + N\left\|u_\gamma\right\|_U^2 + \left\|S(u_\gamma - u_0)\right\|_G^2 \leq c, \tag{1.79}$$

where c is a constant independent of γ.
We deduce from (1.79) that (u_γ) is bounded in U, then we can extract a subsequence still be denoting (u_γ) converges weakly to $u \in U$.
It's clear that for every $v \in U$

$$\mathcal{J}\left(v, g\right) - \mathcal{J}\left(u_0, g\right) - \gamma\left\|g\right\|_G^2 \leq \mathcal{J}\left(v, g\right) - \mathcal{J}\left(u_0, g\right) \ \forall g \in G,$$

hence

$$\mathcal{J}\left(u_\gamma, g\right) - \mathcal{J}\left(u_0, g\right) - \gamma\left\|g\right\|_G^2 \leq \sup_{g\in G}\left(\mathcal{J}\left(v, g\right) - \mathcal{J}\left(u_0, g\right)\right) \ \forall g \in G,$$

when γ tend to 0 we obtain

$$\mathcal{J}\left(u, g\right) - \mathcal{J}\left(u_0, g\right) \leq \sup_{g\in G}\left(\mathcal{J}\left(v, g\right) - \mathcal{J}\left(u_0, g\right)\right) \ \forall g \in G,$$

In conclusion, u is a no-regret control. ■

Low-regret control optimality system

The following proposition gives the optimality system for low-regret control u_γ.

Proposition 1.11 *Low-regret control u_γ, solution of (1.76)-(1.77) is characterized by the unique solution $\{y_\gamma, \zeta_\gamma, \rho_\gamma, p_\gamma\}$ of the following optimality system*

$$\begin{cases} Ay_\gamma = f + Bu_\gamma, \\ A^*\zeta_\gamma = C^*C(y_\gamma - y(0,0)), \\ A\rho_\gamma = \frac{1}{\gamma}\beta\beta^*\zeta_\gamma, \\ A^*p_\gamma = C^*(Cy_\gamma - y_d) + C^*C\rho_\gamma, \\ (B^*p_\gamma + Nu_\gamma, w)_U \geq 0 \quad \forall w \in U. \end{cases} \tag{1.80}$$

where $y(u_\gamma, 0) = y_\gamma$, $\zeta(u_\gamma) = \zeta_\gamma$.

Proof. Let u_γ be solution to (1.76) (1.77). A first order necessary condition gives for every $w \in U$

$$\left(C^*\left(Cy(u_\gamma,0) - y_d\right), y(w,0) - y\left(0,0\right)\right)_{\mathcal{V}} + N\left(u_\gamma, w\right)_U + \frac{1}{\gamma}\left(S(u_\gamma - u_0), S\left(w\right)\right)_G \geq 0. \tag{1.81}$$

By definition we have

$$A^*\zeta_\gamma = C^*C(y_\gamma - y(0,0)),$$

Let $\rho_\gamma = \rho(u_\gamma)$ be the solution of

$$A\rho_\gamma = \frac{1}{\gamma}\beta\beta^*\zeta_\gamma,$$

we introduce the adjoint state $p_\gamma = p(u_\gamma)$ defined by

$$A^*p_\gamma = C^*(Cy_\gamma - y_d) + C^*C\rho_\gamma.$$

To simplify the calculations we take $u_0 = 0$.Then,

$$\begin{aligned} \frac{1}{\gamma}\left(S(u_\gamma), S\left(w\right)\right)_G &= \left(\frac{1}{\gamma}\beta^*\zeta(u_\gamma), \beta^*\zeta(w)\right)_G \\ &= \left(\frac{1}{\gamma}\beta\beta^*\zeta(u_\gamma), \zeta(w)\right)_{\mathcal{V}} \\ &= \left(A\rho_\gamma, \zeta(w)\right)_{\mathcal{V}} \\ &= \left(\rho_\gamma, A^*\zeta(w)\right)_{\mathcal{V}} \\ &= \left(\rho_\gamma, C^*C(y(w,0) - y(0,0))\right)_{\mathcal{V}} \\ &= \left(C^*C\rho_\gamma, (y(w,0) - y(0,0)\right)_{\mathcal{V}} \\ &= (A^*p_\gamma - C^*(Cy_\gamma - y_d), (y(w,0) - y(0,0))_{\mathcal{V}} \\ &= (p_\gamma, A(y(w,0) - y(0,0))_{\mathcal{V}} - (C^*(Cy_\gamma - y_d), (y(w,0) - y(0,0))_{\mathcal{V}}. \end{aligned}$$

Then,

$$\frac{1}{\gamma}\left(S(u_\gamma), S\left(w\right)\right)_G = (p_\gamma, Bw)_{\mathcal{V}} - \left(C^*(Cy_\gamma - y_d), (y(w,0) - y(0,0)\right)_{\mathcal{V}}. \tag{1.82}$$

From (1.81) and (1.82), we deduce the following optimality condition

$$(B^*p_\gamma + Nu_\gamma, w)_U \geq 0 \quad \forall w \in U,$$

■

No-regret control optimality system

First, we solve the problem

$$A\rho = \beta g, \ \forall g \in G \ \rho \in \mathcal{V},$$

then

$$A^*\sigma = C^*C\rho, \ \forall \sigma \in \mathcal{V},$$

and we pose

$$Rg = \beta^*\sigma, \ \forall g \in G \ \sigma \in \mathcal{V}.$$

We assume that

$$\left\|Rg\right\|_{\hat{G}} \geq c\left\|g\right\|_G, \ \forall g \in G, \tag{1.83}$$

where $\hat{G}$ is the complement of G

Theorem 1.8 *[46] We suppose that (1.83) is true, the no-regret control u related to u_0 solution of the problem (1.69) is characterized by the single solution $\{y, \lambda, \rho, p\}$*

$$\begin{cases} Ay = f + Bu, \\ A^*\zeta = C^*Cy\left(u,0\right) - y_d, \\ A\rho = \beta\lambda, \lambda \in G, \\ A^*p = C^*\left(Cy\left(u,0\right) - y_d\right) + C^*C\rho, \\ \left(B^*p + Nu, w\right)_U \geq 0, \ \forall w \in U. \end{cases} \tag{1.84}$$

Proof. From relation (1.79) and Theorem 1.7 the sequence (u_γ) converges weakly in U to u the unique no-regret control related to u_0. The operator B continuous from U in $\mathcal{V}'$, then,

$$Bu_\gamma \rightharpoonup Bu \text{ weakly in } \mathcal{V}'.$$

Now from the optimality system of the proposition 1.11 the sequence (Ay_γ) is bounded in $\mathcal{V}'$ and since A is a isomorphism then

$$Ay_\gamma \rightharpoonup Ay \text{ weakly in } \mathcal{V}'.$$

By passing to the limit in the first equation of the system (1.80), we obtain

$$Ay = f + Bu.$$

We also deduced from proposition 1.11 that $B^*p_\gamma = -Nu$ is bounded in $\mathcal{V}'$. Let R be the operator such that $R(\frac{1}{\gamma}\beta^*\zeta_\gamma) = B^*p_\gamma$ then the hypothesis (1.83), we deduce that $(\frac{1}{\gamma}\beta^*\zeta_\gamma)$ is bounded in G closed sub-space of the Hilbert space F. Therefore,

$$\frac{1}{\gamma}\beta^*\zeta_\gamma \rightharpoonup \lambda \in \hat{G} \subset F.$$

Hence $A\rho_\gamma = \beta\frac{1}{\gamma}\beta^*\zeta_\gamma$ is bounded, and then $\left(\rho_\gamma\right)$ also bounded thanks to the isomorphism of A which implies that

$$A\rho_\gamma \rightharpoonup A\rho,\ \rho \in \mathcal{V}.$$

We have $\left(\rho_\gamma\right)$ and (y_γ) are bounded, we obtain that $A^*\rho_\gamma$ is bounded. Therefore, (p_γ) converges to p, which gives by passing to the limit (1.80) the optimality system (1.84) ■

1.4.5 Application (Optimal control of hyperbolic equation with incomplete data)

Let Ω be an open bounded set of $\mathbb{R}^n$ with smooth boundary Γ, denote by $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T), T > 0$. Consider the following hyperbolic system

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} + Ay = f + Bv & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0 + g_0, \dfrac{\partial y}{\partial t}(x,0) = y_1 + g_1 & \text{in } \Omega. \end{cases} \tag{1.85}$$

Let $\mathcal{V}$, H two spaces with $\mathcal{V} \subset H$ dense and separable, $A \in \mathcal{L}(L^2(0,T;\mathcal{V}), L^2(0,T;\mathcal{V}'))$, $B \in \mathcal{L}(U, L^2(0,T;H))$, $f \in L^2((0,T);H)$ and U is a space of controls. Let F be a Hilbert space with $\mathcal{V} \subset F \subset \mathcal{V}'$. Let G_0 be a closed subspace of $\mathcal{V}$ such that $y_0 \in \mathcal{V}$ and $g_0 \in G_0$ and G_1 be a closed subspace of H such that $y_1 \in H$ and $g_1 \in G_1$. We denote by $\left(y(t,x,g), \dfrac{\partial y}{\partial t}(t,x,g)\right)$ the solution of the equation (1.85), where $g = (g_1, g_2)$ and $G = G_0 \times G_1$.
Let $C \in \mathcal{L}\left(L^2((0,T);\mathcal{V}), \mathcal{Z}\right)$ where $\mathcal{Z}$ is the observation space.
Consider the cost function

$$\mathcal{J}(v,g) = \|Cy(v,g) - y_d\|^2_{\mathcal{Z}} + N\|v\|^2_U, \tag{1.86}$$

where y_d given in $\mathcal{Z}$ and $N > 0$.

No-regret control for the hyperbolic equation with incomplete data

We take in the rest of the section $u_0 = 0$. For these cost functions, we are looking for the no-regrets controls u, verifying (1.69).

Lemma 1.2 *For any $v \in U$, we have*

$$\mathcal{J}(v,g)-\mathcal{J}(0,g)=\mathcal{J}(v,0)-\mathcal{J}(0,0)+2\left(\left\langle -\frac{\partial\zeta}{\partial t}(0),g_0\right\rangle_{G_0',G_0}+\langle\zeta(0),g_1\rangle_{G_1',G_1}\right)\ \forall g_0,g_1\in G_0\times G_1, \tag{1.87}$$

where $\zeta(v)$ be solution of

$$\begin{cases} \dfrac{\partial^2\zeta}{\partial t^2}+A^*\zeta=C^*C\left(y(v,0)-y(0,0)\right) & in\ Q, \\ \zeta(x,0)=0 & on\ \Sigma, \\ \zeta(x,T)=0,\dfrac{\partial\zeta}{\partial t}(x,T)=0 & in\ \Omega, \end{cases} \tag{1.88}$$

where A^* is the adjoint of A.

Proof. We have

$$\mathcal{J}(v,g)-\mathcal{J}(0,g)=\mathcal{J}(v,0)-\mathcal{J}(0,0)+2(C^*C\left(y(v,0)-y(0,0)\right),y(0,g)-y(0,0))_{\mathcal{V}}.$$

Now introduce $\zeta(v)$ definied by (1.88), then

$$\begin{aligned}(C^*C\left(y(v,0)-y(0,0)\right),y(0,g)-y(0,0))_{\mathcal{V}} &= (\frac{\partial^2\zeta}{\partial t^2}+A^*\zeta,y(0,g)-y(0,0))_{\mathcal{V}} \\ &= -\left\langle\frac{\partial\zeta}{\partial t}(0),g_0\right\rangle_{G_0',G_0}+\langle\zeta(0),g_1\rangle_{G_1',G_1},\forall g\in G.\end{aligned}$$

■

Low-regret control

Now,we consider the no-regret control

$$\inf_{v\in U}\left(\sup_{g\in G}\left(\mathcal{J}(v,g)-\mathcal{J}(0,g)\right)\right), \tag{1.89}$$

from (1.87) the problem is equivalent to the following one

$$\inf_{v\in U}\left(\mathcal{J}(v,0)-\mathcal{J}(0,0)+2\sup_{g\in G}\left(\left\langle-\frac{\partial\zeta}{\partial t}(0),g_0\right\rangle_{G_0',G_0}+\langle\zeta(0),g_1\rangle_{G_1',G_1}\right)\right),$$

we relax the problem by adding a quadratic perturbation to get

$$\begin{aligned}&\sup_{g\in G}\left(\mathcal{J}(v,g)-\mathcal{J}(0,g)-\gamma\left\|g\right\|_G\right)\\ =\;&\mathcal{J}(v,0)-\mathcal{J}(0,0)+2\sup_{g\in G}\left(\left(\left\langle-\frac{\partial\zeta}{\partial t}\left(v\right)\left(0\right),g_0\right\rangle_{G_0',G_0}+\left\langle\zeta\left(v\right)\left(0\right),g_1\right\rangle_{G_1',G_1}\right)-\gamma\left\|g\right\|_G\right).\end{aligned}$$

Using the Legendre transform we get

$$\sup_{g\in G}\left(\mathcal{J}(v,g)-\mathcal{J}(0,g)-\gamma\left\|g\right\|_G\right)=\mathcal{J}(v,0)-\mathcal{J}(0,0)+\frac{1}{\gamma}\left(\left\|\zeta\left(v\right)\left(0\right)\right\|_{G_1}^2-\left\|\frac{\partial\zeta}{\partial t}\left(v\right)\left(0\right)\right\|_{G_0}^2\right).$$

It comes down to finding

$$\inf_{v\in U}\mathcal{J}^{\gamma}(v), \tag{1.90}$$

where

$$\mathcal{J}^{\gamma}(v)=\mathcal{J}(v,0)-\mathcal{J}(u_0,0)+\frac{1}{\gamma}\left(\left\|\zeta\left(v\right)\left(0\right)\right\|_{G_1}^2-\left\|\frac{\partial\zeta}{\partial t}\left(v\right)\left(0\right)\right\|_{G_0}^2\right) \tag{1.91}$$

Existence and uniqueness of low-regret control

Theorem 1.9 *Problem (1.90) (1.91) has a single solution u_γ called low-regret control.*

Proof. Demonstration is similar to the one made for the Theorem (1.6) ■

Low-regret control optimality system

The following proposition gives the optimality system for low-regret control u_γ.

Proposition 1.12 *low-regre control u_γ, solution of (1.90) (1.91) is characterized by the unique* *solution* $\{y_\gamma, \zeta_\gamma, \rho_\gamma, p_\gamma\}$ *of the following optimality system*

$$\begin{cases} \dfrac{\partial^2 y_\gamma}{\partial t^2} + Ay_\gamma = f + Bu_\gamma, \\ y(x,0) = y_0, \dfrac{\partial y}{\partial t}(x,0) = y_1, \\ \dfrac{\partial^2 \zeta_\gamma}{\partial t^2} + A^*\zeta_\gamma = C^*C\left(y(u_\gamma,0) - y(0,0)\right), \\ \zeta(x,T) = 0, \dfrac{\partial \zeta}{\partial t}(x,T) = 0, \\ \dfrac{\partial^2 \rho_\gamma}{\partial t^2} + A\rho_\gamma = 0, \\ \rho_\gamma(x,0) = \dfrac{1}{\gamma}\dfrac{\partial \zeta}{\partial t}(0), \dfrac{\partial \rho_\gamma}{\partial t}(x,0) = \dfrac{1}{\gamma}\zeta(0), \\ \dfrac{\partial^2 p_\gamma}{\partial t^2} + A^*p_\gamma = C^*C(y_\gamma - y_d) + C^*C\rho_\gamma, \\ p_\gamma(x,T) = 0, \dfrac{\partial p_\gamma}{\partial t}(x,T) = 0, \\ (B^*p_\gamma + Nu_\gamma, w)_U \geq 0, \ \forall w \in U. \end{cases} \tag{1.92}$$

where $y\left(u_\gamma, 0\right) = y_\gamma,\ \zeta(u_\gamma) = \zeta_\gamma$.

Proof. Let u_γ be solution to (1.90) (1.91). A first order necessary condition gives for every $w \in U$

$$(Cy(u_\gamma,0)-y_d, C\left(y(w,0) - y\left(0,0\right)\right))_{\mathcal{Z}}+N(u_\gamma,w)_U-(\frac{1}{\gamma}\frac{\partial \zeta}{\partial t}\left(u_\gamma\right)(0), \frac{\partial \zeta}{\partial t}\left(w\right)(0))_{\mathcal{V}}+(\frac{1}{\gamma}\zeta\left(u_\gamma\right)(0), \zeta\left(w\right)(0))_{\mathcal{V}} \geq \tag{1.93}$$

Let's introduce a new state given by

$$\begin{cases} \dfrac{\partial^2 \rho_\gamma}{\partial t^2} + A\rho_\gamma = 0 & \text{in } Q, \\ \rho_\gamma\left(x,0\right) = 0 & \text{on } \Sigma, \\ \rho_\gamma(x,0) = \dfrac{1}{\gamma}\dfrac{\partial \zeta}{\partial t}(0), \dfrac{\partial \rho_\gamma}{\partial t}(x,0) = \dfrac{1}{\gamma}\zeta(0) & \text{in } \Omega, \end{cases} \tag{1.94}$$

then (1.93) becomes

$$(C^*Cy(u_\gamma,0) - y_d + C^*C\rho_\gamma, y(w,0) - y\left(0,0\right))_{\mathcal{V}} + N(u_\gamma,w)_U \geq 0. \tag{1.95}$$

We introduce p_γ solution of the problem

$$\begin{cases} \dfrac{\partial^2 p_\gamma}{\partial t^2} + Ap_\gamma = C^*Cy(u_\gamma,0) - y_d + C^*C\rho_\gamma & \text{in } Q, \\ p_\gamma\left(x,0\right) = 0 & \text{on } \Sigma, \\ p_\gamma(x,T) = 0, \dfrac{\partial p_\gamma}{\partial t}(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{1.96}$$

Then, we have

$$\mathcal{J}^{\gamma\prime}(u_\gamma)(w) = (B^*p_\gamma + Nu_\gamma, w)_U \geq 0, \ \forall w \in U, \tag{1.97}$$

■

No-regret control optimality system

Theorem 1.10 *The no-regrets control* u *solution of the problem (1.89) is characterized by the single solution* $\{y, \lambda, \rho, p\}$

$$\begin{cases} \dfrac{\partial^2 y_\gamma}{\partial t^2} + Ay = f + Bu, \\ y(x,0) = y_0, \dfrac{\partial y}{\partial t}(x,0) = y_1, \\ \dfrac{\partial^2 \zeta}{\partial t^2} + A^*\zeta = C^*C\left(y(u,0) - y(0,0)\right), \\ \zeta(x,T) = 0, \dfrac{\partial \zeta}{\partial t}(x,T) = 0, \\ \dfrac{\partial^2 \rho}{\partial t^2} + A\rho = 0, \\ \rho(x,0) = \lambda_0, \dfrac{\partial \rho}{\partial t}(x,0) = \lambda_1, \\ A^*p = C^*C(y - y_d) + C^*C\rho, \\ p(x,T) = 0, \dfrac{\partial p}{\partial t}(x,T) = 0, \\ (B^*p + Nu, w)_U \geq 0 \quad \forall w \in U, \end{cases} \tag{1.98}$$

where

$$\begin{aligned} \lambda_0 &= \lim_{\gamma \to 0} \frac{1}{\gamma} \frac{\partial \zeta}{\partial t}(0) \ \textit{and} \ \lambda_0 \in \hat{G}_0 \ \textit{the complement of} \ G_0. \\ \lambda_1 &= \lim_{\gamma \to 0} \frac{1}{\gamma} \zeta(0) \ \textit{and} \ \lambda_1 \in \hat{G}_1 \ \textit{the complement of} \ G_1. \end{aligned}$$

Chapter 2

Averaged controllability of some parameter dependent hyperbolic systems

In the first part of this chapter, null averaged controllability problems are studied by an extension of the Hilbert Uniqueness Method for parameter dependent wave equation. This approach (HUM), introduced by Lions [43] in 1986, is based on uniqueness theorems leading to the construction of suitable Hilbert spaces of the controllable spaces. Then, in the next part, to treat the problem of parameter dependent vibrating plate equation, we follow the same steps as in the previous part. In the last part, our objective is to extend the notion of regional controllability for parameter dependent hyperbolic systems. We use an approach based on an extension of the Hilbert uniqueness method devoted to the calculation of the control which steers the averaged state(with respect to such a parameter) towards the desired state only on a given part of the system evolution domain.

2.1 Averaged controllability of parameter dependent wave equation

In this section, we consider the averaged null controllability property for wave equation depending on an unknown parameter under the effect of boundary control. We prove the averaged inverse inequality (averaged observability inequality) by using the multiplier method. Then, applying the Hilbert uniqueness method well adapted to design a control chosen independently of the parameter value which steers the average of the state to the origin.

2.1.1 Problem statement

Let an open bounded subset Ω of $\mathbb{R}^n$ with a regular boundary Γ and $T > 0$, we denote Γ_0 a non-empty open part of Γ, $Q = \Omega \times]0,T[\,,\ \Sigma = \Gamma \times]0,T[\,, \Sigma_0 = \Gamma_0 \times]0,T[\,.$ We consider the following wave equation with unknown parameter σ and a boundary control action

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \sigma^2 \Delta y = 0 & \text{in } Q, \\ y = \begin{cases} u \\ 0 \end{cases} & \begin{matrix} \text{on } \Sigma_0, \\ \text{on} \Sigma \setminus \Sigma_0, \end{matrix} \\ y\left(0,x\right) = y_0\left(x\right), \ \dfrac{\partial y}{\partial t}\left(0,x\right) = y_1\left(x\right) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where the velocity of propagation parameter σ is supposed to be unknown in $\left(\sigma_1, \sigma_2\right), \mathcal{V} = L^2\left(\Omega\right) \times H^{-2}\left(\Omega\right)$ is the state space, u presents a boundary control action in $L^2\left(\Sigma_0\right),\ \left(y_0, y_1\right) \in \mathcal{V}$, all are independent of the parameter σ.
It's well known that the wave equation (2.1) has a unique solution $\left(y_u(t,x,\sigma), \dfrac{\partial y_u}{\partial t}(t,x,\sigma)\right)$ in $C\left(0,T; L^2\left(\Omega\right)\right) \cap C^1\left(0,T; H^{-1}\left(\Omega\right)\right)$ [43].
We introduce the following notions of null averaged controllability for the system (2.1).

Definition 2.1 *The system (2.1) is said to be null averaged controllable if there exists a control u independent of the parameter σ such that*

$$\left(\int_{\sigma_1}^{\sigma_2} y_u\left(T,x,\sigma\right) d\sigma, \int_{\sigma_1}^{\sigma_2} \frac{\partial y_u}{\partial t}\left(T,x,\sigma\right) d\sigma\right) = \left(0,0\right). \tag{2.2}$$

However, let's consider the following homogeneous wave equation with smooth initial conditions given by

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial t^2} - \sigma^2 \Delta \phi = 0 & \text{in } Q, \\ \phi = 0 & \text{on } \Sigma, \\ \phi\left(0,x\right) = \phi_0\left(x\right), \ \dfrac{\partial \phi}{\partial t}\left(0,x\right) = \phi_1\left(x\right) & \text{in } \Omega, \end{cases} \tag{2.3}$$

where the initial data $\left(\phi_0, \phi_1\right) \in H_0^1\left(\Omega\right) \times L^2\left(\Omega\right)$ are independent of the parameter σ. It's well known that (2.3) has a unique solution [43].
Let's define for all $t \in (0,T)$ the averaged energy with respect to σ associated to (2.3) by

$$E_a\left(t\right) = \frac{1}{2} \int_{\sigma_1}^{\sigma_2} \int_{\Omega} \left[\left|\frac{\partial \phi}{\partial t}\right|^2 + \sigma^2 \left|\nabla \phi\right|^2\right] dx d\sigma. \tag{2.4}$$

Lemma 2.1 *Let $\phi = \phi(t,x,\sigma)$ be a solution to (2.3). Then the averaged energy (2.4) is conserved, i.e. for all $t \in (0,T)$*

$$E_a(t) = E_a(0) = \frac{1}{2} \int_{\sigma_1}^{\sigma_2} \int_{\Omega} \left[\left|\phi_1\right|^2 + \sigma^2 \left|\nabla \phi_0\right|^2\right)\right] dx d\sigma. \tag{2.5}$$

Proof. Multiply (2.3) by $\dfrac{\partial \phi}{\partial t}$, integrate on $(\sigma_1, \sigma_2) \times Q$, then we apply Green formula, Fubini theorem (See Appendix), we obtain

$$\begin{aligned}
0 &= \int_{\sigma_1}^{\sigma_2} \int_\Omega \left(\frac{\partial^2 \phi}{\partial t^2} - \sigma^2 \Delta \phi \right) \left(\frac{\partial \phi}{\partial t} \right) dx d\sigma \\
&= \int_{\sigma_1}^{\sigma_2} \int_\Omega \frac{1}{2} \frac{\partial}{\partial t} \left| \frac{\partial \phi}{\partial t} \right|^2 - \sigma^2 \Delta \phi \frac{\partial \phi}{\partial t} dx d\sigma \\
&= \int_{\sigma_1}^{\sigma_2} \left(\frac{1}{2} \int_\Omega \frac{\partial}{\partial t} \left| \frac{\partial \phi}{\partial t} \right|^2 dx + \sigma^2 \int_\Omega \nabla \phi . \nabla \left(\frac{\partial \phi}{\partial t} \right) dx - \sigma^2 \int_\Gamma \nabla \phi . \eta \frac{\partial \phi}{\partial t} d\Gamma \right) d\sigma \\
&= \int_{\sigma_1}^{\sigma_2} \left(\frac{1}{2} \int_\Omega \frac{\partial}{\partial t} \left| \frac{\partial \phi}{\partial t} \right|^2 dx + \sigma^2 \int_\Omega \frac{\partial}{\partial t} \left| \nabla \phi \right|^2 dx - \sigma^2 \int_\Gamma \nabla \phi . \eta \frac{\partial \phi}{\partial t} d\Gamma \right) d\sigma.
\end{aligned}$$

Considering the boundary conditions we get

$$\frac{\partial E_a(t)}{\partial t} = 0,$$

and as a result, we have energy conservation. ■

2.1.2 Averaged inverse & direct inequalities

The main objective of this main section is to establish averaged inverse inequality and averaged direct inequality of the problem (2.3), such inequalities would be very important in the application of Hilbert Uniqueness Method (HUM) exactly to prove the coercivity and consistency of an operator who plays a key role.

For ease of notation, the convention of recurring indices will be applied in the remainder of this chapter, such as

$$q_k \eta_k = \sum_{k=1}^{n} q_k \eta_k.$$

Lemma 2.2 *Let $q : \overline{\Omega} \to \mathbb{R}^n$ be a vector field of class C^1 independent of the parameter σ, then for every ϕ solution for (2.3), we have the following identity*

$$\begin{aligned}
\tfrac{1}{2} \int_{\sigma_1}^{\sigma_2} \int_\Sigma \sigma^2 \left| \frac{\partial \phi}{\partial \eta} \right|^2 q_k \eta_k d\Sigma d\sigma &= \int_{\sigma_1}^{\sigma_2} \int_\Omega \frac{\partial \phi}{\partial t} q_k \frac{\partial \phi}{\partial x_k} dx \Big|_0^T d\sigma \\
&+ \tfrac{1}{2} \int_{\sigma_1}^{\sigma_2} \int_Q \frac{\partial q_k}{\partial x_k} \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - \sigma^2 \left| \nabla \phi \right|^2 \right) dx dt d\sigma \\
&+ \int_{\sigma_1}^{\sigma_2} \int_Q \sigma^2 \frac{\partial \phi}{\partial x_i} \frac{\partial q_k}{\partial x_i} \frac{\partial \phi}{\partial x_k} dx dt d\sigma.
\end{aligned} \tag{2.6}$$

Proof. We multiply the equation (2.3) with $q_k \dfrac{\partial \phi}{\partial x_k}$ and we integrate in $(\sigma_1, \sigma_2) \times Q$ we obtain

$$\underbrace{\int_{\sigma_1}^{\sigma_2}\int_Q \frac{\partial^2 \phi}{\partial t^2} q_k \frac{\partial \phi}{\partial x_k} dxdtd\sigma}_{(I)} - \underbrace{\int_{\sigma_1}^{\sigma_2}\int_Q \sigma^2 \Delta \phi q_k \frac{\partial \phi}{\partial x_k} dxdtd\sigma}_{(II)} = 0.$$

First, analysis of (I)

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2}\int_Q \frac{\partial^2 \phi}{\partial t^2} q_k \frac{\partial \phi}{\partial x_k} dxdtd\sigma \\ = &\int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial \phi}{\partial t} q_k \frac{\partial \phi}{\partial x_k} dx\Big|_0^T d\sigma - \int_{\sigma_1}^{\sigma_2}\int_Q q_k \frac{\partial \phi}{\partial t}\frac{\partial}{\partial x_k}\left(\frac{\partial \phi}{\partial t}\right) dxdtd\sigma,\end{aligned}$$

and we know that

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2}\int_Q \frac{\partial \phi}{\partial t} q_k \frac{\partial}{\partial x_k}\left(\frac{\partial \phi}{\partial t}\right) dxdtd\sigma \\ = &\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q q_k \frac{\partial}{\partial x_k}\left|\frac{\partial \phi}{\partial t}\right|^2 dxdtd\sigma\end{aligned} \tag{2.7}$$

on the other hand, Gauss's divergence formula (See Appendix) gives us

$$\int_\Omega \frac{\partial}{\partial x_k}\left(q_k \left|\frac{\partial \phi}{\partial t}\right|^2\right) dx = \int_\Gamma q_k \left|\frac{\partial \phi}{\partial t}\right|^2 \eta_k dx \tag{2.8}$$

then, it results that

$$-\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q q_k \frac{\partial}{\partial x_k}\left|\frac{\partial \phi}{\partial t}\right|^2 dxdtd\sigma = \frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q \frac{\partial q_k}{\partial x_k}\left|\frac{\partial \phi}{\partial t}\right|^2 dxdtd\sigma \tag{2.9}$$

since $\dfrac{\partial \phi}{\partial t} = 0$ on Σ. From (2.7) and (2.9) it results

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2}\int_Q \frac{\partial^2 \phi}{\partial t^2} q_k \frac{\partial \phi}{\partial x_k} dxdtd\sigma \\ = &\int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial \phi}{\partial t} q_k \frac{\partial \phi}{\partial x_k} dx\Big|_0^T d\sigma + \frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q \frac{\partial q_k}{\partial x_k}\left|\frac{\partial \phi}{\partial t}\right|^2 dxdtd\sigma.\end{aligned} \tag{2.10}$$

Otherwise, analysis of (II), apply the Green formula (See Appendix) to get

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2}\int_Q \sigma^2 \Delta \phi q_k \frac{\partial \phi}{\partial x_k} dxdtd\sigma \\ = &\int_{\sigma_1}^{\sigma_2}\int_\Sigma \sigma^2 \frac{\partial \phi}{\partial \eta} q_k \frac{\partial \phi}{\partial x_k} d\Sigma d\sigma - \int_{\sigma_1}^{\sigma_2}\int_Q \sigma^2 \nabla \phi \cdot \nabla\left(q_k \frac{\partial \phi}{\partial x_k}\right) dxdtd\sigma.\end{aligned}$$

On the other hand, we have

$$\nabla(\phi)\cdot\nabla\left(q_k\frac{\partial\phi}{\partial x_k}\right)=\frac{\partial\phi}{\partial x_i}q_k\frac{\partial^2\phi}{\partial x_i\partial x_k}+\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}$$
$$=\ \frac{1}{2}q_k\frac{\partial}{\partial x_k}\left|\nabla\phi\right|^2+\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k},$$

then,

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\Delta\phi q_k\frac{\partial\phi}{\partial x_k}dxdtd\sigma\\ =\ &\int_{\sigma_1}^{\sigma_2}\int_\Sigma\sigma^2\frac{\partial\phi}{\partial\eta}q_k\frac{\partial\phi}{\partial x_k}d\Sigma d\sigma-\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2q_k\frac{\partial}{\partial x_k}\left|\nabla\phi\right|^2dxdtd\sigma\\ &-\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}dxdtd\sigma,\end{aligned}$$

from Gauss's divergence (See Appendix) we get

$$\begin{aligned}&\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2q_k\frac{\partial}{\partial x_k}\left|\nabla\phi\right|^2dxdtd\sigma\\ =\ &\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_\Sigma\sigma^2q_k\left|\nabla\phi\right|^2\eta_kd\Sigma d\sigma-\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\frac{\partial q_k}{\partial x_k}\left|\nabla\phi\right|^2dxdtd\sigma,\end{aligned}$$

hence

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\Delta\phi q_k\frac{\partial\phi}{\partial x_k}dxdtd\sigma\\ =\ &\int_{\sigma_1}^{\sigma_2}\int_\Sigma\sigma^2\frac{\partial\phi}{\partial\eta}q_k\frac{\partial\phi}{\partial x_k}d\Sigma d\sigma-\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_\Sigma\sigma^2q_k\left|\nabla\phi\right|^2\eta_kd\Sigma d\sigma\\ &+\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\frac{\partial q_k}{\partial x_k}\left|\nabla\phi\right|^2dxdtd\sigma-\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}dxdtd\sigma.\end{aligned}\tag{2.11}$$

Remark that

$$\int_\Sigma\left|\nabla\phi\right|^2q_k\eta_kd\Sigma=\int_\Sigma\left|\frac{\partial\phi}{\partial\eta}\right|^2q_k\eta_kd\Sigma,$$

and (2.11) will have the form

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\Delta\phi q_k\frac{\partial\phi}{\partial x_k}dxdtd\sigma\\ =\ &\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_\Sigma\sigma^2q_k\left|\frac{\partial\phi}{\partial\eta}\right|^2\eta_kd\Sigma d\sigma\\ &+\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\frac{\partial q_k}{\partial x_k}\left|\nabla\phi\right|^2dxdtd\sigma\\ &-\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}dxdtd\sigma.\end{aligned}\tag{2.12}$$

As a result, from (2.12) and (2.10), we deduce that

$$\begin{gathered}\int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial\phi}{\partial t}q_k\frac{\partial\phi}{\partial x_k}dx\Big|_0^T d\sigma\\ +\tfrac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q \frac{\partial q_k}{\partial x_k}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-\sigma^2\left|\nabla\phi\right|^2\right)dxdtd\sigma\\ -\tfrac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_\Sigma \sigma^2\left|\frac{\partial\phi}{\partial\eta}\right|^2 q_k\eta_k d\Sigma d\sigma\\ +\int_{\sigma_1}^{\sigma_2}\int_Q \sigma^2\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}dxdtd\sigma=0,\end{gathered}$$

hence the identity (2.6). ■

Let's give an *averaged direct inequality* which leads to an important property of ϕ will be called *averaged hidden regularity property.*

Theorem 2.1 *(**Averaged direct inequality**) The solution of (2.3) verifies the following inequality*

$$\int_{\sigma_1}^{\sigma_2}\int_\Sigma \sigma^2\left|\frac{\partial\phi}{\partial\eta}(t,x,\sigma)\right|^2 d\Sigma d\sigma\leq CE_a(0)\,, \tag{2.13}$$

where C is a positive constant.

Proof. In (2.6), choose a vector field $q=h$ such that $h.\eta=1$ on Γ, we refer the reader to [39, chapter 1, lemma 3.1], and use the conservation of averaged energy property to deduce easily that

$$\begin{gathered}\tfrac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_\Sigma \sigma^2\left|\frac{\partial\phi}{\partial\eta}d\sigma\right|^2 d\Sigma\leq\|h\|_{L^\infty(\Omega)}\int_{\sigma_1}^{\sigma_2}\left(\frac{\partial\phi}{\partial t},\frac{\partial\phi}{\partial x_k}\right)\Big|_0^T d\sigma\\ +\tfrac{1}{2}\|\nabla h\|_{(L^\infty(\Omega))^n}\int_{\sigma_1}^{\sigma_2}\int_Q\left[\left|\frac{\partial\phi}{\partial t}\right|^2-\sigma^2\left|\nabla\phi\right|^2\right]dxdtd\sigma\\ +\|\nabla h\|_{(L^\infty(\Omega))^n}\int_{\sigma_1}^{\sigma_2}\int_Q\left|\nabla\phi\right|^2dxdtd\sigma\\ \leq c\|h\|_{W^{1,\infty}(\Omega)}\int_{\sigma_1}^{\sigma_2}\int_\Omega\left[\left|\phi_1\right|^2+\sigma^2\left|\nabla\phi_0\right|^2)\right]dxd\sigma\\ \leq CE_a(0)\,.\end{gathered}$$

■

Remark 2.1 *the estimate (2.13) implies the following averaged hidden regularity property for the solution of (2.3)*

$$\frac{\partial\phi}{\partial\eta}(t,x,\sigma)\in L^2(\Sigma)\,. \tag{2.14}$$

The rest of this section will be dedicated to establish and prove the averaged inverse inequality contributing to the main outcomes of uniqueness.

Let us introduce the following notation, for any fixed $x_0\in\mathbb{R}^n$, we set

$$m(x) = x - x_0,\ \forall x \in \mathbb{R}^n,$$
$$\Gamma_0 = \{x \in \Gamma; m(x).\eta(x) > 0\},$$
$$\Sigma_0 = \Gamma_0 \times]0, T[,$$
$$R_0 = \|m(x)\|_{L^\infty(\Omega)},$$
$$T_0 = 2R_0.$$

$\eta(x)$ is a field of unit normal vectors directed outward from Q

Theorem 2.2 *(**Averaged inverse inequality**) Assume that Γ is of class C^2, so for any $T > T_0$ and every solution ϕ of homogeneous problem (2.3), the following inequality is verified*

$$(T - T_0) E_a(0) \leq \frac{R_0}{2} \int_{\sigma_1}^{\sigma_2} \int_{\Sigma_0} \sigma^2 \left| \frac{\partial \phi}{\partial \eta}(t, x, \sigma) \right|^2 d\Sigma d\sigma. \tag{2.15}$$

Proof. With the choice of multipliers $q_k(x) = m_k(x)$, we have

$$\frac{\partial q_k}{\partial x_j} = \delta_{jk}, \sum_{k=0}^{n} \frac{\partial q_k}{\partial x_k} = n, \text{ and } \frac{\partial q_k}{\partial x_j} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_k} = |\nabla \phi|,$$

then, identity (2.6) becomes

$$\begin{aligned} \int_{\sigma_1}^{\sigma_2} \int_\Omega \frac{\partial \phi}{\partial t} m_k \frac{\partial \phi}{\partial x_k} dx \Big|_0^T d\sigma + \frac{n}{2} \int_{\sigma_1}^{\sigma_2} \int_Q \frac{\partial q_k}{\partial x_k} \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - \sigma^2 |\nabla \phi|^2 \right) dx dt d\sigma \\ + \int_{\sigma_1}^{\sigma_2} \int_Q \sigma^2 |\nabla \phi| \, dx dt d\sigma \\ = \frac{1}{2} \int_{\sigma_1}^{\sigma_2} \int_\Sigma \sigma^2 \left| \frac{\partial \phi}{\partial \eta} \right|^2 m_k \eta_k d\Sigma d\sigma. \end{aligned} \tag{2.16}$$

On Σ_0, due the Cauchy-Schwarz inequality (See Appendix), we get

$$0 < m(x).\eta(x) = \sum_{k=0}^{n} m_k.\eta_k \leq \left(\sum_{k=0}^{n} m_k^2 \right)^{\frac{1}{2}} . \left(\sum_{k=0}^{n} \eta_k^2 \right)^{\frac{1}{2}} = \|m(x)\| \leq R_0,$$

therefore, the identity (2.16) becomes

$$\begin{aligned} X + \frac{n}{2} \int_{\sigma_1}^{\sigma_2} \int_Q \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - \sigma^2 |\nabla \phi|^2 \right) dx dt d\sigma \\ + \int_{\sigma_1}^{\sigma_2} \int_Q \sigma^2 |\nabla \phi| \, dx dt d\sigma \\ \leq \frac{R_0}{2} \int_{\sigma_1}^{\sigma_2} \int_\Sigma \sigma^2 \left| \frac{\partial \phi}{\partial \eta} \right|^2 d\Sigma d\sigma, \end{aligned} \tag{2.17}$$

where

$$X = \int_{\sigma_1}^{\sigma_2} \int_\Omega \frac{\partial \phi}{\partial t} m_k \frac{\partial \phi}{\partial x_k} dx d\sigma \Big|_0^T.$$

In addition

$$
\begin{aligned}
& X+\frac{n}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\left(\left|\frac{\partial\phi}{\partial t}\right|^2-\sigma^2\left|\nabla\phi\right|^2\right)dxdtd\sigma+\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\left|\nabla\phi\right|dxdtd\sigma \\
= \;& X+\frac{n}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\left|\frac{\partial\phi}{\partial t}\right|^2dxdtd\sigma-\frac{n}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\left|\nabla\phi\right|^2dxdtd\sigma \\
& +\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\left|\nabla\phi\right|dxdtd\sigma+\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\left|\frac{\partial\phi}{\partial t}\right|^2dxdtd\sigma-\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\left|\frac{\partial\phi}{\partial t}\right|^2dxdtd\sigma \\
= \;& X+\frac{n-1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\left|\frac{\partial\phi}{\partial t}\right|^2dxdtd\sigma+\frac{2-n}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\left|\nabla\phi\right|^2dxdtd\sigma+\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\left|\frac{\partial\phi}{\partial t}\right|^2dxdtd\sigma \\
= \;& X+\frac{n-1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\left|\frac{\partial\phi}{\partial t}\right|^2dxdtd\sigma-\frac{n-1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\left|\nabla\phi\right|^2dxdtd\sigma \\
& +\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\left|\frac{\partial\phi}{\partial t}\right|^2dxdtd\sigma+\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\left|\nabla\phi\right|^2dxdtd\sigma \\
= \;& X+\frac{n-1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\left|\frac{\partial\phi}{\partial t}\right|^2-\sigma^2\left|\nabla\phi\right|^2dxdtd\sigma+\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\left|\frac{\partial\phi}{\partial t}\right|^2+\sigma^2\left|\nabla\phi\right|^2dxdtd\sigma.
\end{aligned}
$$

Let

$$
Y=\int_{\sigma_1}^{\sigma_2}\int_Q\left(\left|\frac{\partial\phi}{\partial t}\right|^2-\sigma^2\left|\nabla\phi\right|^2\right)dxdtd\sigma.
$$

Furthermore, we use energy conservation (2.3) and Fubini theorem (See Appendix), inequality (2.17) becomes

$$
X+\frac{n-1}{2}Y+TE_a(0)\leq\frac{R_0}{2}\int_{\sigma_1}^{\sigma_2}\int_{\Sigma_0}\sigma^2\left|\frac{\partial\phi}{\partial\eta}\right|^2d\Sigma d\sigma \tag{2.18}
$$

We multiply the equation (2.5) by ϕ and we integrate. We get

$$
\int_{\sigma_1}^{\sigma_2}\int_Q\left(\frac{\partial^2\phi}{\partial t^2}-\sigma^2\Delta\phi\right)\phi dxdtd\sigma=0,
$$

we have

$$
\int_{\sigma_1}^{\sigma_2}\int_Q\frac{\partial^2\phi}{\partial t^2}\phi dxdtd\sigma=\int_{\sigma_1}^{\sigma_2}\int_\Omega\frac{\partial\phi}{\partial t}\phi dxd\sigma\Big|_0^T-\int_{\sigma_1}^{\sigma_2}\int_Q\left|\frac{\partial\phi}{\partial t}\right|^2dxdtd\sigma,
$$

on the other hand, according to Green's identity (See Appendix) and the fact that ($\phi=0$ on Σ), we have

$$
\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\Delta\phi\phi dxdtd\sigma=\int_{\sigma_1}^{\sigma_2}\int_\Sigma\sigma^2\phi\frac{\partial\phi}{\partial\eta}d\Sigma d\sigma-\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\left|\nabla\phi\right|^2dxdtd\sigma=-\int_{\sigma_1}^{\sigma_2}\int_Q\sigma^2\left|\nabla\phi\right|^2dxdtd\sigma,
$$

then

$$
\int_{\sigma_1}^{\sigma_2}\int_\Omega\frac{\partial\phi}{\partial t}\phi dxd\sigma\Big|_0^T-\int_{\sigma_1}^{\sigma_2}\int_Q\left(\left|\frac{\partial\phi}{\partial t}\right|^2-\sigma^2\left|\nabla\phi\right|^2\right)dxdtd\sigma=0,
$$

hence

$$Y=\int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial\phi}{\partial t}\phi dxd\sigma\Big|_0^T,$$

thus, we have

$$X+\frac{n-1}{2}Y=\int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial\phi}{\partial t}\left(m_k\frac{\partial\phi}{\partial x_k}+\frac{n-1}{2}\phi\right)dxd\sigma\Big|_0^T \tag{2.19}$$

Cauchy-Schwarz inequality (See Appendix) gives

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2}\int_\Omega \left(\frac{\partial\phi}{\partial t}\left(m_k\frac{\partial\phi}{\partial x_k}+\frac{n-1}{2}\phi\right)\right)dxd\sigma\\ \leq\;& \int_{\sigma_1}^{\sigma_2}\frac{\varepsilon}{2}\int_\Omega\left|\frac{\partial\phi}{\partial t}\right|^2dxd\sigma+\int_{\sigma_1}^{\sigma_2}\frac{1}{2\varepsilon}\int_\Omega\left|m_k\frac{\partial\phi}{\partial x_k}+\frac{n-1}{2}\phi\right|^2dxd\sigma.\end{aligned} \tag{2.20}$$

On the other hand

$$\begin{aligned}&\int_\Omega\left|m_k\frac{\partial\phi}{\partial x_k}+\frac{n-1}{2}\phi\right|^2dx\\ =\;&\int_\Omega\left|m_k\frac{\partial\phi}{\partial x_k}\right|^2dx+\frac{(n-1)^2}{4}\int_\Omega|\phi|^2dx\\ &+(n-1)\int_\Omega m_k\frac{\partial\phi}{\partial x_k}\phi dx.\end{aligned} \tag{2.21}$$

In addition

$$\int_\Omega m_k\frac{\partial\phi}{\partial x_k}\phi dx=\frac{1}{2}\int_\Omega m_k\frac{\partial}{\partial x_k}\left(|\phi|^2\right)dx,$$

according to the Gauss divergence formula (See Appendix) and as ($\phi=0$ on Σ) we have

$$\int_\Omega\frac{\partial}{\partial x_k}\left(m_k|\phi|^2\right)dx=\int_\Gamma\left(m_k\eta_k|\phi|^2\right)d\Gamma=0,$$

$$\frac{1}{2}\int_\Omega m_k\frac{\partial}{\partial x_k}\left(|\phi|^2\right)dx=-\frac{1}{2}\int_\Omega\frac{\partial m_k}{\partial x_k}\left(\phi^2\right)dx=-\frac{n}{2}\int_\Omega|\phi|^2dx. \tag{2.22}$$

Using (2.22) in (2.21) , we obtain

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2}\int_\Omega\left|m_k\frac{\partial\phi}{\partial x_k}+\frac{n-1}{2}\phi\right|^2dxd\sigma\\ =\;&\int_{\sigma_1}^{\sigma_2}\left(\int_\Omega\left|m_k\frac{\partial\phi}{\partial x_k}\right|^2dx+\frac{(n-1)^2}{4}\int_\Omega|\phi|^2dx-\frac{n(n-1)}{2}\int_\Omega|\phi|^2dx\right)d\sigma\\ =\;&\int_{\sigma_1}^{\sigma_2}\left(\int_\Omega\left|m_k\frac{\partial\phi}{\partial x_k}\right|^2dx-\frac{(n^2-1)}{4}\int_\Omega|\phi|^2dx\right)d\sigma\\ \leq\;&\int_{\sigma_1}^{\sigma_2}\int_\Omega\left|m_k\sigma\frac{\partial\phi}{\partial x_k}\right|^2dxd\sigma\\ \leq\;& R_0^2\int_{\sigma_1}^{\sigma_2}\sigma^2\int_\Omega|\nabla\phi|^2dxd\sigma.\end{aligned} \tag{2.23}$$

Substitute (2.23) in (2.20) with the choice $\varepsilon = R_0$, we get

$$\begin{aligned} & \int_{\sigma_1}^{\sigma_2}\int_\Omega \left(\frac{\partial \phi}{\partial t}\left(m_k\frac{\partial \phi}{\partial x_k}+\frac{n-2}{2}\phi\right)\right)dxd\sigma \\ \leq\ & \frac{R_0}{2}\int_{\sigma_1}^{\sigma_2}\int_\Omega \left|\frac{\partial \phi}{\partial t}\right|^2 dxd\sigma + \frac{R_0}{2}\int_{\sigma_1}^{\sigma_2}\int_\Omega \sigma^2\left|\nabla\phi\right|^2 dxd\sigma \\ \leq\ & R_0E_a(0). \end{aligned} \tag{2.24}$$

From (2.26) and (2.24), we deduce that

$$\begin{aligned} \left|X+\frac{n-2}{2}Y\right| &= \left|\int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial \phi}{\partial t}\left(m_k\frac{\partial \phi}{\partial x_k}+\frac{n-2}{2}\phi\right)dxd\sigma\Big|_0^T\right| \\ &\leq 2\sup_{t\in[0,T]}\left|\int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial \phi}{\partial t}\left(m_k\frac{\partial \phi}{\partial x_k}+\frac{n-2}{2}\phi\right)dxd\sigma\right| \\ &\leq 2\left\|\int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial \phi}{\partial t}\left(m_k\frac{\partial \phi}{\partial x_k}+\frac{n-2}{2}\phi\right)dxd\sigma\right\|_{L^\infty(0,T)} \\ &\leq 2R_0E_a(0). \end{aligned}$$

We take $T_0 = 2R_0$, we obtain

$$\left|X+\frac{n-1}{2}Y\right| \leq T_0E_a(0),$$

then

$$\begin{aligned} TE(0)-T_0E(0) &\leq TE(0) - \int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial \phi}{\partial t}\left(m_k\frac{\partial \phi}{\partial x_k}+\frac{n-1}{2}\phi\right)dxd\sigma\Big|_0^T \\ &\leq \frac{R_0}{2}\int_{\sigma_1}^{\sigma_2}\int_{\Sigma_0}\sigma^2\left|\frac{\partial \phi}{\partial \eta}\right|^2 d\Sigma d\sigma. \end{aligned}$$

Finally, we get

$$(T-T_0)\,E(0) \leq \frac{R_0}{2}\int_{\sigma_1}^{\sigma_2}\int_{\Sigma_0}\sigma^2\left|\frac{\partial \phi}{\partial \eta}\right|^2 d\Sigma d\sigma d\Sigma d\sigma.$$

■

2.1.3 Null averaged controllability. Hilbert uniqueness method

The HUM technique is based on certain uniqueness criteria for the homogeneous system (2.3) and on the construction, by completion, of certain Hilbertian spaces suited to the system structure.
In this subsection, we present the main steps devoted to calculating the control $u(t)$ that leads the system's averaged state (2.1) to the null state. The method is based on the Hilbert Uniqueness Method (HUM) introduced by Lions (See [43]).

Theorem 2.3 *Assume that the assumptions of Theorem 2.2 hold. Then for any given initial data* $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ *there exists* $u \in L^2(\Sigma_0)$ *such that the solution of (2.1) satisfies (2.2).*

Proof. Fix $(\phi_0, \phi_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and let's introduce the following backward equation by

$$\begin{cases} \dfrac{\partial^2 \psi}{\partial t^2} - \sigma^2 \Delta \psi = 0 & \text{in } Q, \\ \psi = \begin{cases} \int_{\sigma_1}^{\sigma_2} \sigma^2 \dfrac{\partial \phi}{\partial \eta}(t, x, \sigma)\, d\sigma & \text{on } \Sigma_0, \\ 0 & \text{on } \Sigma \setminus \Sigma_0, \end{cases} \\ \psi(T, x, \sigma) = 0,\ \dfrac{\partial \psi}{\partial t}(T, x, \sigma) = 0 & \text{in } \Omega. \end{cases} \tag{2.25}$$

Multiply (2.3) by ψ and integrate on $[\sigma_1, \sigma_2] \times Q$ to get

$$\begin{aligned} & \int_{\sigma_1}^{\sigma_2} \int_Q (\frac{\partial^2 \phi}{\partial t^2} - \sigma^2 \Delta \phi) \psi dx dt d\sigma \\ = & \int_{\sigma_1}^{\sigma_2} \int_\Omega \frac{\partial \phi}{\partial t}(T, x, \sigma) \psi(T, x, \sigma) d\sigma dx - \int_{\sigma_1}^{\sigma_2} \int_\Omega \frac{\partial \phi}{\partial t}(0, x, \sigma) \psi(0, x, \sigma) d\sigma dx \\ & - \int_{\sigma_1}^{\sigma_2} \int_\Omega \phi(T, x, \sigma) \frac{\partial \psi}{\partial t}(T, x, \sigma) dx d\sigma + \int_{\sigma_1}^{\sigma_2} \int_\Omega \phi(0, x, \sigma) \frac{\partial \psi}{\partial t}(0, x, \sigma) dx d\sigma \\ & + \int_{\sigma_1}^{\sigma_2} \int_\Sigma \sigma^2 \psi(t, x, \sigma) \frac{\partial \phi}{\partial \eta}(t, x, \sigma)\, d\Sigma d\sigma - \int_{\sigma_1}^{\sigma_2} \int_\Sigma \sigma^2 \phi(t, x, \sigma) \frac{\partial \psi}{\partial \eta}(t, x, \sigma)\, d\Sigma d\sigma \\ = & \ 0. \end{aligned}$$

Consider boundary condition to obtain

$$\int_\Omega \phi_0(x) \int_{\sigma_1}^{\sigma_2} \frac{\partial \psi}{\partial t}(0, x, \sigma)\, d\sigma dx - \int_\Omega \phi_1(x) \int_{\sigma_1}^{\sigma_2} \psi(0, x, \sigma)\, d\sigma dx = \int_{\Sigma_0} \left| \int_{\sigma_1}^{\sigma_2} \sigma^2 \frac{\partial \phi}{\partial \eta} d\sigma \right|^2 d\Sigma.$$

Define an operator Λ by

$$\Lambda(\phi_0, \phi_1) = \left(\int_{\sigma_1}^{\sigma_2} \frac{\partial \psi}{\partial t}(0, x, \sigma)\, d\sigma, -\int_{\sigma_1}^{\sigma_2} \psi(0, x, \sigma)\, d\sigma \right),$$

therefore,

$$(\Lambda(\phi_0, \phi_1), (\phi_0, \phi_1)) = \int_{\Sigma_0} \left| \int_{\sigma_1}^{\sigma_2} \sigma^2 \frac{\partial \phi}{\partial \eta} d\sigma \right|^2 d\Sigma. \tag{2.26}$$

By Theorem (2.1), Theorem (2.2) and the Lax-Milgram theorem, it follows that Λ defines an isomorphism from $H_0^1(\Omega) \times L^2(\Omega)$ to $H^{-1}(\Omega) \times L^2(\Omega)$. That is for all $(y_1, -y_0) \in H^{-1}(\Omega) \times L^2(\Omega)$, the equation $\Lambda(\phi_0, \phi_1) = (y_1, -y_0)$ has a unique solution (ϕ_0, ϕ_1). With this initial condition, we solve (2.3) and then we solve (2.25). Thus, we have found a control

$$u = \int_{\sigma_1}^{\sigma_2} \sigma^2 \frac{\partial \phi}{\partial \eta}(t, x, \sigma)\, d\sigma,$$

such that the solution of (2.1) satisfies (2.2). ■

2.2 Averaged controllability of parameter dependent vibrating plate equation

In this section, we use the same steps used in the previous section to treat the problem of the vibrating plate equation i.e. we demonstrate an averaged inverse and direct inequalities giving some coercivity and continuity results for the main introduced operator in Hilbert uniqueness method and moreover, we describe the main steps of the Hilbert uniqueness method for the null averaged controllability problem.

2.2.1 Problem statement

Let Ω be a non-empty bounded domain in $\mathbb{R}^n$ having a regular boundary Γ and $T > 0$, we denote $Q = \Omega \times]0, T[\,$, $\Sigma = \Gamma \times]0, T[$. We consider the following controlled system which describes the vibrations of plate

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} + \Delta(a(x,\sigma)\Delta y) = 0 & \text{in } Q, \\ y = 0, & \text{on } \Sigma, \\ \dfrac{\partial y}{\partial \eta} = \begin{cases} u & \\ 0 & \end{cases} & \begin{matrix} \text{on } \Sigma_0, \\ \text{on} \Sigma \setminus \Sigma_0, \end{matrix} \\ y(0,x) = y_0(x),\ \dfrac{\partial y}{\partial t}(0,x) = y_1(x) & \text{in } \Omega, \end{cases} \tag{2.27}$$

where $a \in C^1(]\sigma_1, \sigma_2[\,, L^\infty(\Omega)), \mathcal{V} = L^2(\Omega) \times H^{-2}(\Omega)$ is the state spase, u presents a boundary control action in $L^2(\Sigma_0)$. For $(y_0, y_1) \in L^2(\Omega) \times H^{-2}(\Omega)$ and u (all are independent of the parameter σ), system (2.27) has a unique weak solution $\left(y_u(t,x,\sigma), \dfrac{\partial y_u}{\partial t}(t,x,\sigma)\right) \in C(0,T;\mathcal{V})$ [31].

We are interested in the following controllability problem

Definition 2.2 *For $T > 0$ and the initial data $(y_0, y_1) \in L^2(\Omega) \times H^{-2}(\Omega)$, the system (2.27) is said to be null averaged controllable if there exists a control u independent of the parameter σ such that*

$$\left(\int_{\sigma_1}^{\sigma_2} y(T,x,\sigma)\, d\sigma, \int_{\sigma_1}^{\sigma_2} \frac{\partial y}{\partial t}(T,x,\sigma)\, d\sigma\right) = (0,0). \tag{2.28}$$

Let us now introduce the following homogeneous plate system

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial t^2} + \Delta(a(x,\sigma)\Delta \phi) = 0 & \text{in } Q, \\ \phi = \dfrac{\partial \phi}{\partial \eta} = 0 & \text{on } \Sigma, \\ \phi(0,x) = \phi_0(x),\ \dfrac{\partial \phi}{\partial t}(0,x) = \phi_1(x) & \text{in } \Omega, \end{cases} \tag{2.29}$$

where $(\phi_0, \phi_1) \in H_0^2(\Omega) \times L^2(\Omega)$, in prinsiple, are independent of the parametre σ. It's well known that (2.29) has a unique solution [70].

We define the averaged energy $E_a(t)$ with respect to σ associated to the system (2.29) defined by the following formula

$$E_a(t) = \frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_{\Omega}\left[\left|\frac{\partial\phi}{\partial t}\right|^2 + a(x,\sigma)\,|\Delta\phi|^2\right] dx d\sigma. \tag{2.30}$$

Lemma 2.3 *For all $\phi = \phi(x,t,\sigma)$ solution of the problem (2.29) the averaged energy (2.30) is conserved for all $t \in (0,T)$ i.e.*

$$E_a(t) = E_a(0) = \frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_{\Omega}\left[|\phi_1|^2 + a(x,\sigma)\,|\Delta\phi_0|^2\right] dx d\sigma. \tag{2.31}$$

Proof. We multipling the equation (2.29) by $\dfrac{\partial\phi}{\partial t}$ and integrate on $(\sigma_1,\sigma_2)\times Q$, then we apply Green formula and Fubini theorem (See Appendix), we obtain

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2}\int_{\Omega}\left(\frac{\partial^2\phi}{\partial t^2} + \Delta(a(x,\sigma)\Delta\phi)\right)\frac{\partial\phi}{\partial t} dx d\sigma\\
=\;& \frac{1}{2}\int_{\sigma_1}^{\sigma_2}\frac{\partial}{\partial t}\int_{\Omega}\left|\frac{\partial\phi}{\partial t}\right|^2 dx d\sigma + \int_{\sigma_1}^{\sigma_2}\int_{\Omega} a(x,\sigma)\Delta\phi\Delta\left(\frac{\partial\phi}{\partial t}\right) dx d\sigma\\
&-\int_{\sigma_1}^{\sigma_2}\int_{\Sigma}\left[a(x,\sigma)\Delta\phi\left(\frac{\partial}{\partial\eta}\left(\frac{\partial\phi}{\partial t}\right)\right) - \frac{\partial}{\partial\eta}\left(a(x,\sigma)\Delta\phi\right)\frac{\partial\phi}{\partial t}\right] d\Sigma d\sigma.
\end{aligned}$$

Considering the boundary conditions we get

$$\frac{\partial E_a(t)}{\partial t} = 0,$$

and as a result, we have energy conservation. ■

2.2.2 Averaged direct & inverse inequalities

The results of this section will serve as a basis for the Hilbert Uniqueness Method (HUM) for parameter dependent vibrating plate equation. We shall here establish an identity for the weak solutions of the problem (2.29) which will then prove the averaged observability theorems. These will be obtained by using a special multiplier.

Lemma 2.4 *Let $q : \overline{\Omega} \to \mathbb{R}^n$ be a vector field of class C^2 independent of the uncertainty parameter*

σ, then for every solution ϕ of (2.29), the following identity holds

$$\begin{aligned}
&\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_{\Sigma} a(x,\sigma)q_k\eta_k\left|\Delta\phi\right|^2 d\Sigma d\sigma \\
= &\int_{\sigma_1}^{\sigma_2}\int_{\Omega}\frac{\partial\phi}{\partial t}q_k\frac{\partial\phi}{\partial x_k}dxd\sigma\Big|_0^T+\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_{Q}\frac{\partial q_k}{\partial x_k}\left|\frac{\partial\phi}{\partial t}\right|^2 dxdtd\sigma \\
&+\int_{\sigma_1}^{\sigma_2}\int_{Q} a(x,\sigma)\Delta q_k\Delta\phi\frac{\partial\phi}{\partial x_k}dxdtd\sigma \\
&+2\int_{\sigma_1}^{\sigma_2}\int_{Q} a(x,\sigma)\frac{\partial q_k}{\partial x_j}\Delta\phi\frac{\partial^2\phi}{\partial x_j\partial x_k}dxdtd\sigma \\
&-\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_{Q}\frac{\partial}{\partial x_k}\left(a(x,\sigma)q_k\right)\left|\Delta\phi\right|^2 dxdtd\sigma.
\end{aligned} \tag{2.32}$$

Proof. We multiply the equation (2.29) with $q_k\dfrac{\partial\phi}{\partial x_k}$ and we integrate in $(\sigma_1,\sigma_2)\times Q$ we obtain

$$\underbrace{\int_{\sigma_1}^{\sigma_2}\int_{Q}\frac{\partial^2\phi}{\partial t^2}q_k\frac{\partial\phi}{\partial x_k}dxdtd\sigma}_{(I)}+\underbrace{\int_{\sigma_1}^{\sigma_2}\int_{Q}\Delta(a(x,\sigma)\Delta\phi)q_k\frac{\partial\phi}{\partial x_k}dxdtd\sigma}_{(II)}=0.$$

First, analysis of (I)

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2}\int_{Q}\frac{\partial^2\phi}{\partial t^2}q_k\frac{\partial\phi}{\partial x_k}dxdtd\sigma \\
= &\int_{\sigma_1}^{\sigma_2}\int_{\Omega}\frac{\partial\phi}{\partial t}q_k\frac{\partial\phi}{\partial x_k}dx\Big|_0^T d\sigma-\int_{\sigma_1}^{\sigma_2}\int_{Q}\frac{\partial\phi}{\partial t}q_k\frac{\partial}{\partial x_k}\left(\frac{\partial\phi}{\partial t}\right)dxdtd\sigma,
\end{aligned} \tag{2.33}$$

and we know that

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2}\int_{Q}\frac{\partial\phi}{\partial t}q_k\frac{\partial}{\partial x_k}\left(\frac{\partial\phi}{\partial t}\right)dxdtd\sigma \\
= &\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_{Q}q_k\frac{\partial}{\partial x_k}\left|\frac{\partial\phi}{\partial t}\right|^2 dxdtd\sigma \\
= &-\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_{Q}\frac{\partial q_k}{\partial x_k}\left|\frac{\partial\phi}{\partial t}\right|^2 dxdtd\sigma,
\end{aligned} \tag{2.34}$$

since $\dfrac{\partial\phi}{\partial t}=0$ on Σ. From (2.33) and (2.34) it results

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2}\int_{Q}\frac{\partial^2\phi}{\partial t^2}q_k\frac{\partial\phi}{\partial x_k}dxdtd\sigma \\
= &\int_{\sigma_1}^{\sigma_2}\int_{\Omega}\frac{\partial\phi}{\partial t}q_k\frac{\partial\phi}{\partial x_k}dxd\sigma\Big|_0^T+\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_{Q}\frac{\partial q_k}{\partial x_k}\left|\frac{\partial\phi}{\partial t}\right|^2 dxdtd\sigma.
\end{aligned} \tag{2.35}$$

Otherwise, analysis of (II)

apply the Green formula (See Appendix) to get

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2}\int_Q \Delta(a(x,\sigma)\Delta\phi) q_k \frac{\partial\phi}{\partial x_k} dxdtd\sigma \\
=&\int_{\sigma_1}^{\sigma_2}\int_Q a(x,\sigma)\Delta\phi\Delta\left(q_k\frac{\partial\phi}{\partial x_k}\right) dxdtd\sigma \\
&+\int_{\sigma_1}^{\sigma_2}\int_\Sigma \frac{\partial}{\partial\eta}\left(a(x,\sigma)\Delta\phi\right) q_k\frac{\partial\phi}{\partial x_k} d\Sigma d\sigma \\
&-\int_{\sigma_1}^{\sigma_2}\int_\Sigma a(x,\sigma)\Delta\phi\frac{\partial}{\partial\eta}\left(q_k\frac{\partial\phi}{\partial x_k}\right) d\Sigma d\sigma \\
=&\int_{\sigma_1}^{\sigma_2}\int_Q a(x,\sigma)\Delta\phi\Delta\left(q_k\frac{\partial\phi}{\partial x_k}\right) dxdtd\sigma \\
&-\int_{\sigma_1}^{\sigma_2}\int_\Sigma a(x,\sigma)\Delta\phi\frac{\partial}{\partial\eta}\left(q_k\frac{\partial\phi}{\partial x_k}\right) d\Sigma d\sigma, \qquad (2.36)
\end{aligned}$$

because $q_k\dfrac{\partial\phi}{\partial x_k} = 0$ on Σ. In addition

$$\int_{\sigma_1}^{\sigma_2} \Delta\left(q_k\frac{\partial\phi}{\partial x_k}\right) d\sigma = \int_{\sigma_1}^{\sigma_2}\left(\Delta q_k\frac{\partial\phi}{\partial x_k} + 2\frac{\partial q_k}{\partial x_j}\frac{\partial^2\phi}{\partial x_j\partial x_k} + q_k\frac{\partial\Delta\phi}{\partial x_k}\right) d\sigma, \qquad (2.37)$$

and

$$\int_{\sigma_1}^{\sigma_2}\frac{\partial}{\partial\eta}\left(q_k\frac{\partial\phi}{\partial x_k}\right) d\sigma = \int_{\sigma_1}^{\sigma_2}\left(\frac{\partial q_k}{\partial\eta}\frac{\partial\phi}{\partial x_k} + q_k\frac{\partial^2\phi}{\partial\eta\partial x_k}\right) d\sigma = \int_{\sigma_1}^{\sigma_2} q_k\frac{\partial^2\phi}{\partial\eta\partial x_k} d\sigma \text{ on } \Sigma, \qquad (2.38)$$

therefore

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2}\int_Q \Delta(a(x,\sigma)\Delta\phi) q_k\frac{\partial\phi}{\partial x_k} dxdtd\sigma \\
=&\int_{\sigma_1}^{\sigma_2}\int_Q a(x,\sigma)\Delta\phi\Delta q_k\frac{\partial\phi}{\partial x_k} dxdtd\sigma \\
&+2\int_{\sigma_1}^{\sigma_2}\int_Q a(x,\sigma)\Delta\phi\frac{\partial q_k}{\partial x_j}\frac{\partial^2\phi}{\partial x_j\partial x_k} dxdtd\sigma \\
&+\int_{\sigma_1}^{\sigma_2}\int_Q a(x,\sigma)\Delta\phi q_k\frac{\partial\Delta\phi}{\partial x_k} dxdtd\sigma \\
&-\int_{\sigma_1}^{\sigma_2}\int_\Sigma a(x,\sigma)\Delta\phi q_k\frac{\partial^2\phi}{\partial\eta\partial x_k} d\Sigma d\sigma. \qquad (2.39)
\end{aligned}$$

We note that

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2}\int_Q a(x,\sigma)\Delta\phi q_k\frac{\partial\Delta\phi}{\partial x_k} dxdtd\sigma \\
=&\ \frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q a(x,\sigma)q_k\frac{\partial}{\partial x_k}\left|\Delta\phi\right|^2 dxdtd\sigma \\
=&-\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\frac{\partial}{\partial x_k}\left(a(x,\sigma)q_k\right)\left|\Delta\phi\right|^2 dxdtd\sigma + \frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_\Sigma a(x,\sigma)q_k\eta_k\left|\Delta\phi\right|^2 d\Sigma d\sigma, \qquad (2.40)
\end{aligned}$$

then

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2}\int_Q \Delta(a(x,\sigma)\Delta\phi)q_k\frac{\partial\phi}{\partial x_k}dxdtd\sigma\\
= &\int_{\sigma_1}^{\sigma_2}\int_Q a(x,\sigma)\Delta\phi\Delta q_k\frac{\partial\phi}{\partial x_k}dxdtd\sigma\\
&+2\int_{\sigma_1}^{\sigma_2}\int_Q a(x,\sigma)\Delta\phi\frac{\partial q_k}{\partial x_j}\frac{\partial^2\phi}{\partial x_j\partial x_k}dxdtd\sigma\\
&-\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q \frac{\partial}{\partial x_k}\left(a(x,\sigma)q_k\right)|\Delta\phi|^2 dxdtd\sigma\\
&-\int_{\sigma_1}^{\sigma_2}\int_\Sigma\left(a(x,\sigma)\Delta\phi q_k\frac{\partial^2\phi}{\partial\eta\partial x_k}-\frac{1}{2}a(x,\sigma)q_k\eta_k|\Delta\phi|^2\right)d\Sigma d\sigma. \qquad (2.41)
\end{aligned}$$

Furthermore, since $\phi\in H_0^2(\Omega)$, we have

$$\frac{\partial^2\phi}{\partial\eta\partial x_k}=\frac{\partial^2\phi}{\partial\eta^2}\eta_k \text{ and } \frac{\partial^2\phi}{\partial^2 x_k}=\frac{\partial^2\phi}{\partial\eta^2}\eta_k^2 \text{ on } \Sigma, \qquad (2.42)$$

hence

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2}\int_\Sigma\left(-a(x,\sigma)\Delta\phi q_k\frac{\partial^2\phi}{\partial\eta\partial x_k}+\frac{1}{2}a(x,\sigma)q_k\eta_k|\Delta\phi|^2\right)d\Sigma d\sigma\\
= &-\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_\Sigma a(x,\sigma)q_k\eta_k|\Delta\phi|^2 d\Sigma d\sigma. \qquad (2.43)
\end{aligned}$$

As a result, from (2.35) and (2.41)- (2.43), we deduce that

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial\phi}{\partial t}q_k\frac{\partial\phi}{\partial x_k}d\sigma dx\Big|_0^T d\sigma+\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\frac{\partial q_k}{\partial x_k}\left|\frac{\partial\phi}{\partial t}\right|^2 dxdtd\sigma\\
&+\int_{\sigma_1}^{\sigma_2}\int_Q a(x,\sigma)\Delta q_k\Delta\phi\frac{\partial\phi}{\partial x_k}dxdtd\sigma\\
&+2\int_{\sigma_1}^{\sigma_2}\int_Q a(x,\sigma)\frac{\partial q_k}{\partial x_j}\Delta\phi\frac{\partial^2\phi}{\partial x_j\partial x_k}dxdtd\sigma\\
&-\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_Q\frac{\partial}{\partial x_k}\left(a(x,\sigma)q_k\right)|\Delta\phi|^2 dxdtd\sigma\\
&-\int_{\sigma_1}^{\sigma_2}\frac{1}{2}\int_\Sigma a(x,\sigma)q_k\eta_k|\Delta\phi|^2 d\Sigma d\sigma\\
= &\ 0, \qquad (2.44)
\end{aligned}$$

hence the identity (2.32). ■

Theorem 2.4 *(**Averaged direct inequality**) Given $T>0$ arbitrarily, there exists a constant $C=C(T)>0$ s.t the solution of (2.29) satisfies the following inequality*

$$\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_\Sigma a(x,\sigma)|\Delta\phi|^2 d\Sigma d\sigma\le CE_a(0). \qquad (2.45)$$

Proof. We apply the identity in Lemma 2.4 with a vector field $q = h$ and $h.\eta = 1$ constructed in [43, **chapter1, lemma 3.2**] we obtain easily the estimate (2.45) with a suitable constant C, where C is a constant depending on $\|h\|_{W^{2,\infty}(\Omega)}$. ■

Remark 2.2 *From Theorem 2.4 we deduce the following trace result*

$$(\phi_0, \phi_1) \in H_0^2(\Omega) \times L^2(\Omega) \Rightarrow \Delta\phi \in L^2(\Sigma).$$

Now, we serve a basis result for the next subsection (Hilbert Uniqueness Method (HUM)). This result is averaged observability inequality.

Let us introduce the following notation, for any fixed $x_0 \in \mathbb{R}^n$, we set

$$\begin{aligned}
&m(x) = x - x_0,\ \forall x \in \mathbb{R}^n,\\
&\Gamma_0 = \{x \in \Gamma; m(x).\eta(x) > 0\},\\
&\Sigma_0 = \Gamma_0 \times]0, T[,\\
&R_0 = \sup_{x\in\overline{\Omega}} |m(x)|,\\
&T_0 = \frac{R_0^2}{\inf\limits_{\sigma\in[\sigma_1,\sigma_2]} (\lambda_1^2)},
\end{aligned}$$

where $\eta(x)$ is a field of unit normal vectors directed outward from Q and λ_1 is the first eigenvalue of the problem

$$\Delta(a(x,\sigma)\Delta w) = -\lambda_1^2 a(x,\sigma)\Delta w, \qquad \forall w \in H_0^2(\Omega).$$

Theorem 2.5 *(**Averaged inverse inequality**)Assume that Γ is of class C^3, so for any $T > T_0$ and every solution ϕ of homogeneous problem (2.29), the following inequality is verified:*

$$(T - T_0) E_a(0) \le \frac{R_0}{4} \int_{\sigma_1}^{\sigma_2} \int_{\Sigma_0} a(x,\sigma) |\Delta\phi|^2 \, d\Sigma d\sigma. \tag{2.46}$$

Proof. With the choice of multipliers $q_k(x) = m_k(x)$, we have

$$\frac{\partial q_k}{\partial x_j} = \delta_{jk}, \sum_{k=0}^{n} \frac{\partial q_k}{\partial x_k} = n, \text{ and } \frac{\partial q_k}{\partial x_j}\frac{\partial^2\phi}{\partial x_j \partial x_k} = |\triangle\phi|,$$

then, identity (2.32) becomes

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial\phi}{\partial t} m_k \frac{\partial\phi}{\partial x_k} dx d\sigma \Big|_0^T + \frac{n}{2}\int_{\sigma_1}^{\sigma_2}\int_Q \left|\frac{\partial\phi}{\partial t}\right|^2 - a(x,\sigma)|\Delta\phi|^2 dx dt d\sigma\\
&+2\int_{\sigma_1}^{\sigma_2}\int_Q a(x,\sigma)|\Delta\phi|^2 dx dt d\sigma\\
=\ &\frac{1}{2}\int_{\sigma_1}^{\sigma_2}\int_\Sigma a(x,\sigma) m_k \eta_k |\Delta\phi|^2 d\Sigma d\sigma.
\end{aligned} \tag{2.47}$$

On Σ_0, due the Cauchy-Schwarz inequality (See Appendix), we get

$$0 < m(x).\eta(x) = \sum_{k=0}^{n} m_k.\eta_k \leq \left(\sum_{k=0}^{n} m_k^2\right)^{\frac{1}{2}} . \left(\sum_{k=0}^{n} \eta_k^2\right)^{\frac{1}{2}} = \|m(x)\| \leq R_0,$$

therefore, the identity (2.47) becomes

$$\begin{aligned} & X + \frac{n}{2}\int_{\sigma_1}^{\sigma_2}\int_Q \left|\frac{\partial \phi}{\partial t}\right|^2 - a(x,\sigma)\left|\Delta\phi\right|^2 dxdtd\sigma \\ & +2\int_{\sigma_1}^{\sigma_2}\int_Q a(x,\sigma)\left|\Delta\phi\right|^2 dxdtd\sigma \\ \leq \; & \frac{R_0}{2}\int_{\sigma_1}^{\sigma_2}\int_{\Sigma_0} a(x,\sigma)\left|\Delta\phi\right|^2 d\Sigma d\sigma, \end{aligned} \tag{2.48}$$

where

$$X = \int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial \phi}{\partial t} q_k \frac{\partial \phi}{\partial x_k} dxd\sigma\Big|_0^T.$$

In addition

$$\begin{aligned} & X + \frac{n}{2}\int_{\sigma_1}^{\sigma_2}\int_Q \left|\frac{\partial \phi}{\partial t}\right|^2 - a(x,\sigma)\left|\Delta\phi\right|^2 dxdtd\sigma + 2\int_{\sigma_1}^{\sigma_2}\int_Q a(x,\sigma)\left|\Delta\phi\right|^2 dxdtd\sigma \\ = \; & X + \frac{n-2}{2}\int_{\sigma_1}^{\sigma_2}\int_Q \left|\frac{\partial \phi}{\partial t}\right|^2 - a(x,\sigma)\left|\Delta\phi\right|^2 dxdtd\sigma + \int_{\sigma_1}^{\sigma_2}\int_Q \left|\frac{\partial \phi}{\partial t}\right|^2 + a(x,\sigma)\left|\Delta\phi\right|^2 dxdtd\sigma. \end{aligned}$$

Let

$$Y = \int_{\sigma_1}^{\sigma_2}\int_Q \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - a(x,\sigma)\left|\Delta\phi\right|^2\right) dxdtd\sigma.$$

Furthermore, we use energy conservation (2.31) gives and Fubini theorem (See Appendix), inequality (2.48) becomes

$$X + \frac{n-2}{2}Y + 2TE_a(0) \leq \frac{R_0}{2}\int_{\sigma_1}^{\sigma_2}\int_{\Sigma_0} a(x,\sigma)\left|\Delta\phi\right|^2 d\Sigma d\sigma. \tag{2.49}$$

We multiply the equation (2.29) by ϕ and we integrate. We get

$$Y = \int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial \phi}{\partial t}\phi dxd\sigma\Big|_0^T.$$

thus, we have

$$X + \frac{n-2}{2}Y = \int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial \phi}{\partial t}\left(m_k\frac{\partial \phi}{\partial x_k} + \frac{n-2}{2}\phi\right) dxd\sigma\Big|_0^T. \tag{2.50}$$

Cauchy-Schwarz inequality (See Appendix) gives

$$\begin{aligned} & \int_{\sigma_1}^{\sigma_2}\int_\Omega \left(\frac{\partial \phi}{\partial t}\left(m_k\frac{\partial \phi}{\partial x_k} + \frac{n-2}{2}\phi\right)\right) dxd\sigma \\ \leq \; & \int_{\sigma_1}^{\sigma_2}\frac{\varepsilon}{2}\int_\Omega \left|\frac{\partial \phi}{\partial t}\right|^2 dxd\sigma + \int_{\sigma_1}^{\sigma_2}\frac{1}{2\varepsilon}\int_\Omega \left|m_k\frac{\partial \phi}{\partial x_k} + \frac{n-2}{2}\phi\right|^2 dxd\sigma, \end{aligned} \tag{2.51}$$

on the other hand

$$\begin{aligned}
&\int_\Omega \left| m_k \frac{\partial \phi}{\partial x_k} + \frac{n-2}{2}\phi \right|^2 dx \\
= &\int_\Omega \left| m_k \frac{\partial \phi}{\partial x_k} \right|^2 dx + \frac{(n-2)^2}{4} \int_\Omega |\phi|^2 dx \\
&+ (n-2) \int_\Omega m_k \frac{\partial \phi}{\partial x_k} \phi dx. \qquad (2.52)
\end{aligned}$$

In addition

$$\int_\Omega m_k \frac{\partial \phi}{\partial x_k} \phi dx = \frac{1}{2} \int_\Omega m_k \frac{\partial}{\partial x_k} \left(|\phi|^2\right) dx,$$

according to the Gauss divergence formula and as ($\phi = 0$ on Σ), we have

$$\int_\Omega \frac{\partial}{\partial x_k} \left(m_k |\phi|^2\right) dx = \int_\Gamma \left(m_k \eta_k |\phi|^2\right) dx = 0,$$

$$\frac{1}{2} \int_\Omega m_k \frac{\partial}{\partial x_k} \left(|\phi|^2\right) dx = -\frac{1}{2} \int_\Omega \frac{\partial m_k}{\partial x_k} \left(\phi^2\right) dx = -\frac{n}{2} \int_\Omega |\phi|^2 dx. \qquad (2.53)$$

Using (2.53) in (2.52) , we obtain

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2} \int_\Omega \left| m_k \frac{\partial \phi}{\partial x_k} + \frac{n-2}{2}\phi \right|^2 dx d\sigma \\
= &\int_{\sigma_1}^{\sigma_2} \left(\int_\Omega \left| m_k \frac{\partial \phi}{\partial x_k} \right|^2 dx + \frac{(n-2)^2}{4} \int_\Omega |\phi|^2 dx - \frac{n(n-2)}{2} \int_\Omega |\phi|^2 dx \right) d\sigma \\
= &\int_{\sigma_1}^{\sigma_2} \left(\int_\Omega \left| m_k \frac{\partial \phi}{\partial x_k} \right|^2 dx - \frac{(n^2-4)}{4} \int_\Omega |\phi|^2 dx \right) d\sigma \\
\leq &\int_{\sigma_1}^{\sigma_2} \int_\Omega \left| m_k \frac{\partial \phi}{\partial x_k} \right|^2 dx d\sigma \leq R_0^2 \int_{\sigma_1}^{\sigma_2} \int_\Omega |\nabla \phi|^2 dx d\sigma \leq R_0^2 \int_{\sigma_1}^{\sigma_2} \int_\Omega \frac{1}{\lambda_1^2} a(x,\sigma) |\Delta \phi|^2 dx d\sigma \\
\leq &\int_{\sigma_1}^{\sigma_2} \int_\Omega \frac{R_0^2}{(\lambda_1^2)} a(x,\sigma) |\Delta \phi|^2 dx d\sigma. \qquad (2.54)
\end{aligned}$$

Substitute (2.54) in (2.51) with the choice $\varepsilon = \frac{R_0}{\lambda_1}$, we get

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2} \int_\Omega \left(\frac{\partial \phi}{\partial t} \left(m_k \frac{\partial \phi}{\partial x_k} + \frac{n-2}{2}\phi \right) \right) dx d\sigma \\
\leq &\int_{\sigma_1}^{\sigma_2} \int_\Omega \frac{R_0}{2\lambda_1} \left| \frac{\partial \phi}{\partial t} \right|^2 dx d\sigma + \int_{\sigma_1}^{\sigma_2} \int_\Omega \frac{R_0}{2\lambda_1} a(x,\sigma) |\Delta \phi|^2 dx d\sigma \\
\leq &\frac{R_0}{\inf\limits_{\sigma \in [\sigma_1, \sigma_2]} (\lambda_1)} E_a(0) \qquad (2.55)
\end{aligned}$$

From (2.50) and (2.55), we deduce that

$$\begin{aligned}
\left|X+\frac{n-2}{2}Y\right| &= \left|\int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial \phi}{\partial t}\left(m_k\frac{\partial \phi}{\partial x_k}+\frac{n-2}{2}\phi\right)dxd\sigma\Big|_0^T\right| \\
&\leq 2\sup_{t\in[0,T]}\left|\int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial \phi}{\partial t}\left(m_k\frac{\partial \phi}{\partial x_k}+\frac{n-2}{2}\phi\right)dxd\sigma\right| \\
&\leq 2\left\|\int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial \phi}{\partial t}\left(m_k\frac{\partial \phi}{\partial x_k}+\frac{n-2}{2}\phi\right)dxd\sigma\right\|_{L^\infty(0,T)} \\
&\leq 2\frac{R_0}{\inf\limits_{\sigma\in[\sigma_1,\sigma_2]}(\lambda_1)}E_a(0).
\end{aligned}$$

We take $T_0=\dfrac{R_0}{\inf\limits_{\sigma\in[\sigma_1,\sigma_2]}(\lambda_1)}$, we obtain

$$\left|X+\frac{n-2}{2}Y\right|\leq T_0E_a(0),$$

then

$$\begin{aligned}
TE(0)-T_0E(0) &\leq TE(0)-\int_{\sigma_1}^{\sigma_2}\int_\Omega \frac{\partial \phi}{\partial t}\left(m_k\frac{\partial \phi}{\partial x_k}+\frac{n-2}{2}\phi\right)dxd\sigma\Big|_0^T \\
&\leq \frac{R_0}{4}\int_{\sigma_1}^{\sigma_2}\int_{\Sigma_0} a(x,\sigma)\left|\Delta\phi\right|^2 d\Sigma d\sigma.
\end{aligned}$$

Finally, we get

$$(T-T_0)\,E(0)\leq \frac{R_0}{4}\int_{\sigma_1}^{\sigma_2}\int_{\Sigma_0} a(x,\sigma)\left|\Delta\phi\right|^2 d\Sigma d\sigma.$$

■

2.2.3 Null averaged controllability. Hilbert uniqueness method

In this section, we present the essential steps devoted to the calculation of the control $u(t)$ which steers the averaged state of the system (2.27) to the null state. The method is based on the Hilbert Uniqueness Method (HUM) introduced by Lions.

Theorem 2.6 *Assume that the assumptions of Theorem 2.5 hold. Then for any given initial data $(y_0,y_1)\in L^2(\Omega)\times H^{-2}(\Omega)$ there exists $u\in L^2(\Sigma_0)$ such that the solution of (2.27) satisfies (2.28).*

Proof. Fix $(\phi_0, \phi_1) \in H_0^2(\Omega) \times L^2(\Omega)$ arbitrarily. Consider the problem (2.29) and the following backward system

$$\begin{cases} \dfrac{\partial^2\psi}{\partial t^2} + \Delta(a(x,\sigma)\Delta\psi) = 0 & \text{in } Q, \\ \psi = 0 & \text{on } \Sigma, \\ \dfrac{\partial\psi}{\partial\eta} = \begin{cases} \int_{\sigma_1}^{\sigma_2} a(x,\sigma)\Delta\phi d\sigma & \text{on } \Sigma(x_0), \\ 0 & \text{on } \Sigma^*(x_0), \end{cases} \\ \psi(T,x) = \dfrac{\partial\psi}{\partial t}(T,x) = 0 & \text{in } \Omega. \end{cases} \tag{2.56}$$

Define the operator Λ by

$$\Lambda(\phi_0, \phi_1) = \left(\int_{\sigma_1}^{\sigma_2} \frac{\partial\psi}{\partial t}(0,x,\sigma)d\sigma, -\int_{\sigma_1}^{\sigma_2} \psi(0,x,\sigma)d\sigma \right). \tag{2.57}$$

Multiply (2.29) by ψ and integrate on $[\sigma_1, \sigma_2] \times Q$ to obtain

$$\begin{aligned} & \int_{\sigma_1}^{\sigma_2} \int_Q \frac{\partial^2\phi}{\partial t^2} + \Delta(a(x,\sigma)\Delta\phi)\psi dxdtd\sigma \\ = & \int_{\sigma_1}^{\sigma_2} \int_\Omega \frac{\partial\phi}{\partial t}(T,x,\sigma)\psi(T,x,\sigma)d\sigma dx - \int_{\sigma_1}^{\sigma_2} \int_\Omega \frac{\partial\phi}{\partial t}(0,x,\sigma)\psi(0,x,\sigma)d\sigma dx \\ & - \int_{\sigma_1}^{\sigma_2} \int_\Omega \phi(T,x,\sigma)\frac{\partial\psi}{\partial t}(T,x,\sigma)dxd\sigma + \int_{\sigma_1}^{\sigma_2} \int_\Omega \phi(0,x,\sigma)\frac{\partial\psi}{\partial t}(0,x,\sigma)dxd\sigma \\ & + \int_{\sigma_1}^{\sigma_2} \int_\Sigma \psi\frac{\partial}{\partial\eta}\left(a(x,\sigma)\Delta\phi\right) d\Sigma d\sigma - \int_{\sigma_1}^{\sigma_2} \int_\Sigma a(x,\sigma)\Delta\phi\frac{\partial\psi}{\partial\eta}d\Sigma d\sigma \\ = & \ 0. \end{aligned}$$

By boundary condition, we get

$$\int_\Omega \phi_0(x) \int_{\sigma_1}^{\sigma_2} \frac{\partial\psi}{\partial t}(0,x,\sigma)d\sigma dx - \int_\Omega \phi_1(x) \int_{\sigma_1}^{\sigma_2} \psi(0,x,\sigma)d\sigma dx = \int_{\Sigma_0} \left| \int_{\sigma_1}^{\sigma_2} a(x,\sigma)\Delta\phi d\sigma \right|^2 d\Sigma,$$

therefore

$$\left(\Lambda(\phi_0,\phi_1), (\phi_0,\phi_1)\right) = \int_{\Sigma_0} \left| \int_{\sigma_1}^{\sigma_2} a(x,\sigma)\Delta\phi d\sigma \right|^2 d\Sigma. \tag{2.58}$$

By Theorem (2.4), Theorem (2.5) and the Lax-Milgram theorem, it follows that Λ defines an isomorphism from $H_0^2(\Omega) \times L^2(\Omega)$ to $H^{-2}(\Omega) \times L^2(\Omega)$. That is for all $(y_1, -y_0) \in H^{-2}(\Omega) \times L^2(\Omega)$, the equation $\Lambda(\phi_0, \phi_1) = (y_1, -y_0)$ has a unique solution (ϕ_0, ϕ_1). With this initial condition we solve (2.29). Then,we solve (2.56). Thus, we have found a control

$$u = \int_{\sigma_1}^{\sigma_2} a(x,\sigma)\Delta\phi d\sigma,$$

such that the solution of (2.27) satisfies (2.28). ■

2.3 Regional averaged controllability of parameter dependent hyperbolic systems

This section concerns regional averaged controllability of parameter dependent hyperbolic systems. We explore an approach using an extension of the Hilbert Uniqueness Method that leads to the calculation of the control (independent of the parameter) with minimum energy for the cases of internal zone actuator and internal pointwise actuator which drives the system to a given regional averaged state. We consider the case where the subregion of interest is a part of the system evolution domain.

2.3.1 Problem statement

Let an open bounded subset Ω of $\mathbb{R}^n$ with a regular boundary Γ and $T > 0$. We denote ω a subregion of Ω, $Q = \Omega \times]0,T[$, $\Sigma = \Gamma \times]0,T[$. We consider the following hyperbolic PDE depending on an unknown parameter σ with an internal control action

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = A(\sigma)y + Bu & \text{in } Q, \\ y(t,\xi) = 0 & \text{on } \Sigma, \\ y(0,x) = y_0(x), \dfrac{\partial y}{\partial t}(0,x) = y_1(x) & \text{in } \Omega, \end{cases} \tag{2.59}$$

where $A(\sigma)$ is a second-order elliptic linear operator depends on the uncertainty parameter $\sigma \in (0,1)$,$\mathcal{V} = L^2(\Omega) \times L^2(\Omega)$ is the state space, $B \in \mathcal{L}(U, L^2(0,T;\mathcal{V}'))$ is the control operator supposed to be independent of σ where $U = L^2(0,T;\mathbb{R}^p)$ is a space of controls, $u = u(t,x) \in U$ is a distributed control which doesn't depend on σ. The initial data $(y_0(x), y_1(x) \in \mathcal{V}$ are independent of the parameter σ. We denote by $\left(y_u(t,x,\sigma), \dfrac{\partial y_u}{\partial t}(t,x,\sigma)\right)$ the solution of the equation (2.59). The system (2.59) may be written as

$$\begin{cases} \dfrac{\partial z}{\partial t}(t) = \bar{A}(\sigma)z + \bar{B}u(t), \\ z(0,x) = z_0(x) = \begin{pmatrix} y_0(x) \\ y_1(x) \end{pmatrix}, \end{cases} \tag{2.60}$$

where

$$z = \begin{pmatrix} y(t,x,\sigma) \\ \dfrac{\partial y}{\partial t}(t,x,\sigma) \end{pmatrix}, \ \bar{A}(\sigma) = \begin{pmatrix} 0 & I \\ A(\sigma) & 0 \end{pmatrix} \text{ and } \bar{B}u = \begin{pmatrix} 0 \\ Bu(t,x) \end{pmatrix}, \tag{2.61}$$

the solution of (2.60) is given by

$$z(t,\sigma) = \bar{S}(t,\sigma)z_0 + \int_0^t \bar{S}(t-\tau,\sigma)\bar{B}u(\tau)d\tau, \tag{2.62}$$

where $\left\{\bar{S}(t,\sigma)\right\}_{t\geq 0}$is the semi-group generated by $\bar{A}(\sigma)$. For ω being a region of the domain Ω , consider the restriction operator χ_ω

$$\chi_\omega : L^2(\Omega) \times L^2(\Omega) \to L^2(\omega) \times L^2(\omega),$$

$$(z_1, z_2) \mapsto (z_1, z_2)\,|_\omega, \tag{2.63}$$

where χ_ω^* denotes its adjoint defined from $H_0^1(\omega) \times L^2(\omega)$ to $H_0^1(\Omega) \times L^2(\Omega)$ and given by

$$\chi_\omega^*(z_1, z_2) = \begin{cases} (z_1, z_2) & \text{in } \omega, \\ 0 & \text{in } \Omega \backslash \omega. \end{cases} \tag{2.64}$$

2.3.2 Exact and weak regional averaged controllability: Definitions and properties

The *regional averaged controllability* notion is obtained by combining the concept of regional controllability introduced by El Jai in [16] and averaged controllability notion introduced by Zuazua in [69]. Then, we are only interested in steering the state average (with respect to such a parameter) towards the desired state on the subregion ω.
Here, The concept of regional averaged controllability will be formulated and we will give its definitions and properties.

Definition 2.3 *The system (2.59) is said to be* ***exactly regionally averaged controllable*** *on ω if for every final target $(y_d^1, y_d^2) \in L^2(\omega) \times L^2(\omega)$, there exists a control $u \in U$ independent of the parameter σ such that*

$$\chi_\omega\left(\int_0^1 y_u(T,\sigma)d\sigma, \int_0^1 \frac{\partial y_u}{\partial t}(T,\sigma)d\sigma\right) = \left(y_d^1, y_d^2\right). \tag{2.65}$$

Definition 2.4 *The system (2.59) is said to be* ***weakly regionally averaged controllable*** *on ω if for every final target $(y_d^1, y_d^2) \in L^2(\omega) \times L^2(\omega)$ and for all $\varepsilon > 0$, there exists a control $u \in U$ independent of the parameter σ such that*

$$\left\|\int_0^1 y_u(T,\sigma)d\sigma - y_d^1\right\|_{H_0^1(\omega)} + \left\|\int_0^1 \frac{\partial y_u}{\partial t}(T,\sigma)d\sigma - y_d^2\right\|_{L^2(\omega)} \leq \varepsilon. \tag{2.66}$$

Suppose that $\mathcal{H}_A : U \to L^2(\Omega) \times L^2(\Omega)$ is defined by

$$u \mapsto \left(y_u(T,\sigma), \frac{\partial y_u}{\partial t}(T,\sigma)\right). \tag{2.67}$$

It is clear that the system (2.59) is exactly (resp. weakly) regionally averaged controllable on ω if

$$\operatorname{Im}\left(\chi_\omega \int_0^1 \mathcal{H}_A d\sigma\right) = L^2(\omega) \times L^2(\omega), \tag{2.68}$$

$$(resp.\overline{\operatorname{Im}\left(\chi_\omega \int_0^1 \mathcal{H}_A d\sigma\right)} = L^2(\omega) \times L^2(\omega)), \tag{2.69}$$

Remark 2.3

1. *The above definition of regional average controllability is weaker than the standard one (regional controllability) since only the averaged state is transferred and not the state itself .*
2. *The mentioned definition means that we are interested in the transfer of the averaged state with respect to the unknown parameter to the desired function only on the subregion $\omega \subset \Omega$.*

Proposition 2.1 1. *The system (2.59) is exactly (resp.weakly) regionally averaged controllable on ω iff*

$$\ker(\chi_\omega) + \operatorname{Im}\left(\int_0^1 \mathcal{H}_A d\sigma\right) = L^2(\Omega) \times L^2(\Omega). \tag{2.70}$$

2. *The system (2.59) is weakly regionally averaged controllable on ω iff*

$$\ker(\chi_\omega) + \overline{\operatorname{Im}\left(\int_0^1 \mathcal{H}_A d\sigma\right)} = L^2(\Omega) \times L^2(\Omega)). \tag{2.71}$$

Proof. 1. Let $z \in L^2(\Omega) \times L^2(\Omega)$ then $\chi_\omega z \in L^2(\omega) \times L^2(\omega)$. The system (2.59) is exactly regionally averaged controllable on ω, then there exist $u \in U$ such that $\chi_\omega z = \chi_\omega \int_0^1 \mathcal{H}_A u d\sigma$. Let $z_1 = z - \int_0^1 \mathcal{H}_A u d\sigma$ and $z_2 = \int_0^1 \mathcal{H}_A u d\sigma$, we have $z = z_1 + z_2$ with $z_1 \in \ker(\chi_\omega)$ and $z_2 \in \operatorname{Im}\left(\int_0^1 \mathcal{H}_A d\sigma\right)$.
Conversely, let $z \in L^2(\omega) \times L^2(\omega)$, then $\hat{z} = \chi_\omega^* z \in L^2(\Omega) \times L^2(\Omega)$ which allows us to write $\hat{z} = z_1 + z_2$ with $z_1 \in \ker(\chi_\omega)$ and $z_2 \in \operatorname{Im}\left(\int_0^1 \mathcal{H}_A d\sigma\right)$, therefore, $\hat{z} = z_1 + \operatorname{Im}\left(\int_0^1 \mathcal{H}_A u d\sigma\right)$ which gives $z = \chi_\omega \int_0^1 \mathcal{H}_A u d\sigma$ and thus the system (2.59) is exactly regionally averaged controllable on ω.
2. Let $z \in L^2(\Omega) \times L^2(\Omega)$ then $\chi_\omega z \in L^2(\omega) \times L^2(\omega)$. The system (2.59) is weakly regionally averaged controllable on ω, then there exist $u_n \in U$ such that $\chi_\omega z = \lim\left(\chi_\omega \int_0^1 \mathcal{H}_A u_n d\sigma\right)$. Let $z_1 = z - z_2$ with $z_2 = \lim\left(\int_0^1 \mathcal{H}_A u_n d\sigma\right)$, then we have $z = z_1 + z_2$ with $z_1 \in \ker(\chi_\omega)$ and $z_2 \in \overline{\operatorname{Im} \int_0^1 \mathcal{H}_\sigma d\sigma}$.
Conversely, let $z \in L^2(\omega) \times L^2(\omega)$, then $\hat{z} = \chi_\omega^* z \in L^2(\Omega) \times L^2(\Omega)$ which allows us to write $\hat{z} = z_1 + z_2$ with $z_1 \in \ker(\chi_\omega)$ and $z_2 \in \overline{\operatorname{Im} \int_0^1 \mathcal{H}_A d\sigma}$. Consequently, there exists $u_n \in U$ such that $z_2 = \overline{\operatorname{Im} \int_0^1 \mathcal{H}_A u_n d\sigma}$, therefore, $\hat{z} = z_1 + \overline{\operatorname{Im} \int_0^1 \mathcal{H}_A u_n d\sigma}$ which gives $z = \lim\left(\chi_\omega \int_0^1 \mathcal{H}_A u_n d\sigma\right)$, thus the system (2.59) is weakly regionally averaged controllable on ω. ∎

2.3.3 Characterization of regional averaged control

In this subsection, for two different control actions, zone and pointwise, we determine the minimum energy control that assures the transformation of the averaged solution to a desired state in a subregion ω. The technique built here is based on an extension of the Hilbert uniqueness method (HUM) initiated by Lions in [39].

The case of internal zone actuator

We consider the folowing hyperbolic system excited by controls which applied via internal zone actuators (D, f) where $D \subset \Omega$ is the support of the actuator and $f \in L^2(\Omega)$ is the spatial distribution of the action on the support D

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = A(\sigma)y + \chi_D f u & \text{in } Q, \\ y(t,\xi) = 0 & \text{on } \Sigma, \\ y(0,x) = y_0(x), \dfrac{\partial y}{\partial t}(0,x) = y_1(x) & \text{in } \Omega, \end{cases} \tag{2.72}$$

Consider the following minimization problem

$$\begin{cases} \min\limits_{u \in L^2(0,T)} \mathcal{J}(u) = \|u\|^2_{L^2(0,T)} \text{ such that:} \\ \left(\int_0^1 y_u(T,\sigma) d\sigma = y_d^1 \text{ and } \int_0^1 \dfrac{\partial y_u}{\partial t}(T,\sigma) d\sigma = y_d^2 \right) \text{ in } \omega, \end{cases} \tag{2.73}$$

where $\left(y_u(T,\sigma), \dfrac{\partial y_u}{\partial t}(T,\sigma) \right)$ the solution of (2.59), $(y_d^1, y_d^2) \in L^2(\omega) \times L^2(\omega)$ is a desired state at time T.

Next, we show a direct approach to the solution of the regional averaged controllability problem with minimum control energy by using the HUMs.

Let G be a set given by

$$G = \left\{ (\varphi_1, \varphi_0) \in L^2(\Omega) \times L^2(\Omega) \text{ such that } \varphi_1 = \varphi_0 = 0 \text{ in } \Omega \backslash \omega \right\}.$$

In addition, for any $(\varphi_1, \varphi_0) \in G$, we introduce the parameter dependent adjoint system

$$\begin{cases} \dfrac{\partial^2 \varphi}{\partial t^2} = A^*(\sigma)\varphi & \text{in } Q, \\ \varphi(t,\xi) = 0 & \text{on } \Sigma, \\ \varphi(T,x) = \varphi_0(x), \dfrac{\partial \varphi}{\partial t}(T,\sigma) = \varphi_1(x) & \text{in } \Omega, \end{cases} \tag{2.74}$$

where the data at the final time $t = T$, $(\varphi_1, \varphi_0) \in G$ are independent of the uncertainty parameter σ.

We define the following seminorm in G

$$\|(\varphi_1,\varphi_0)\|_G = \left(\int_0^T \left|\left\langle f, \int_0^1 \varphi(t,\sigma)d\sigma\right\rangle_{L^2(D)}\right|^2 dt\right)^{\frac{1}{2}}. \tag{2.75}$$

Lemma 2.5 *The seminorm (2.75) defines a norm on G if the system (2.72) is weakly regionally averaged controllable on ω.*

Proof. The system (2.72) may be written as

$$\frac{\partial z}{\partial t}(t) = \bar{A}(\sigma)z + \bar{B}u(t),$$

where

$$z = \begin{pmatrix} y(t,x,\sigma) \\ \frac{\partial y}{\partial t}(t,x,\sigma)\end{pmatrix},\ \bar{A}(\sigma) = \begin{pmatrix} 0 & I \\ A(\sigma) & 0 \end{pmatrix} \text{ and } \bar{B}u(t,x) = \begin{pmatrix} 0 \\ \chi_D f u\end{pmatrix}.$$

Also, the system (2.74) is equivalent to

$$\begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} \varphi \\ \frac{\partial \varphi}{\partial t}\end{pmatrix} = \bar{A}^*(\sigma)\begin{pmatrix} \varphi \\ \frac{\partial \varphi}{\partial t}\end{pmatrix}, \\ \begin{pmatrix} \varphi(T,x) \\ \frac{\partial \varphi}{\partial t}(T,x)\end{pmatrix} = \begin{pmatrix} \varphi_0(x) \\ \varphi_1(x)\end{pmatrix}, \end{cases} \tag{2.76}$$

where $\bar{A}^*(\sigma)$ is the adjoint operator of $\bar{A}(\sigma)$. For

$$\begin{aligned} \mathcal{H}_\sigma \quad &: \quad U \to L^2(\Omega)\times L^2(\Omega), \\ u \quad &\mapsto \quad \left(y_u(T,\sigma), \frac{\partial y_u}{\partial t}(T,\sigma)\right). \end{aligned} \tag{2.77}$$

Moreover, if the system (2.72) is weakly regionally averaged controllable on ω, we have

$$\overline{\operatorname{Im}\left(\chi_\omega \int_0^1 \mathcal{H}_\sigma d\sigma\right)} = L^2(\omega)\times L^2(\omega) \Leftrightarrow \ker\left(\chi_\omega \int_0^1 \mathcal{H}_\sigma d\sigma\right)^* = \{0\} \Leftrightarrow \ker\left(\int_0^1 \mathcal{H}_\sigma^* d\sigma \chi_\omega^*\right) = \{0\},$$

and we have

$$\mathcal{H}_\sigma^* = \bar{B}^*\bar{S}^*(T-\cdot,\sigma),$$

hence

$$\int_0^1 \mathcal{H}_\sigma^* d\sigma \chi_\omega^* = \int_0^1 \bar{B}^*\bar{S}^*(T-\cdot,\sigma)d\sigma\chi_\omega^*,$$

where $\bar{B}^*$is the adjoint of $\bar{B}$ and $\left(\bar{S}^*(t,\sigma)\right)_{t\geq 0}$ is the semigroup generated by $\bar{A}^*(\sigma)$. The eqution $\|(\varphi_1,\varphi_0)\|_G = 0$ gives $\left\langle f, \int_0^1 \varphi(t,\sigma)d\sigma\right\rangle_{L^2(D)} = 0$ on $[0,T]$, then, we have

$$\left\langle \begin{pmatrix} \chi_D f \\ 0 \end{pmatrix}, \begin{pmatrix} \int_0^1 \varphi(t,\sigma)d\sigma \\ \int_0^1 \frac{\partial \varphi}{\partial t}(t,\sigma)d\sigma \end{pmatrix}\right\rangle_{L^2(\Omega)\times L^2(\Omega)} = 0 \Longleftrightarrow \bar{B}^*\begin{pmatrix} \int_0^1 \varphi(t,\sigma)d\sigma \\ \int_0^1 \frac{\partial \varphi}{\partial t}(t,\sigma)d\sigma \end{pmatrix} = 0.$$

See [64] and [11] for mor details.
In this case the function

$$\begin{pmatrix} \varphi(t,\sigma) \\ \dfrac{\partial \varphi}{\partial t}(t,\sigma) \end{pmatrix} = \bar{S}^*(T-\cdot,\sigma)\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix},$$

is the solution of the system (2.76). Thus, we get

$$\left\langle f, \chi_D \int_0^1 \varphi(t,\sigma)d\sigma \right\rangle_{L^2(\Omega)} = 0 \Leftrightarrow \int_0^1 \bar{B}^*\bar{S}^*(T-\cdot,\sigma)d\sigma \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} = 0.$$

Consequently, $\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \in \ker\left(\int_0^1 \mathcal{H}_\sigma^* d\sigma \chi_\omega^*\right)$. As the system (2.76) is weakly regionally averaged controllable on ω, it follows that $\varphi_0(x) = \varphi_1(x) = 0$ and therefore, (2.75) is a norm. ■

Now, consider the following system

$$\begin{cases} \dfrac{\partial^2 \psi}{\partial t^2} = A(\sigma)\psi + \left\langle f, \int_0^1 \varphi(t,\sigma)d\sigma \right\rangle_{L^2(D)} \chi_D f & \text{in } Q, \\ \psi(t,\xi,\sigma) = 0 & \text{on } \Sigma, \\ \psi(0,x) = y_0(x), \dfrac{\partial \psi}{\partial t}(0,x) = y_1(x) & \text{in } \Omega, \end{cases} \tag{2.78}$$

which is controlled by the solution of the system (2.74).
Let $\mathcal{M}$ be the affine operator defined by

$$\mathcal{M}(\varphi_1,\varphi_0) = \mathcal{P}\left(\int_0^1 \psi(T,\sigma)d\sigma, -\int_0^1 \frac{\partial \psi}{\partial t}(T,\sigma)d\sigma\right), \tag{2.79}$$

where $\mathcal{P} = \chi_\omega^*\chi_\omega$.
The system (2.78) can be decomposed into two subsystems, the first is given by

$$\begin{cases} \dfrac{\partial^2 \psi_1}{\partial t^2} = A(\sigma)\psi_1 & \text{in } Q, \\ \psi_1(t,\xi) = 0 & \text{on } \Sigma, \\ \psi_1(0,x) = y_0(x), \dfrac{\partial \psi_1}{\partial t}(0,x) = y_1(x) & \text{in } \Omega, \end{cases} \tag{2.80}$$

and the second is given by

$$\begin{cases} \dfrac{\partial^2 \psi_2}{\partial t^2} = A(\sigma)\psi_2 + \left\langle f, \int_0^1 \varphi(t,\sigma)d\sigma \right\rangle_{L^2(D)} \chi_D f & \text{in } Q, \\ \psi_2(t,\xi) = 0 & \text{on } \Sigma, \\ \psi_2(0,x) = 0, \dfrac{\partial \psi_2}{\partial t}(0,x) = 0 & \text{in } \Omega. \end{cases} \tag{2.81}$$

Now we consider the operator Λ defined by

$$\Lambda(\varphi_1,\varphi_0) = \mathcal{P}\left(\int_0^1 \psi_2(T,\sigma)d\sigma, -\int_0^1 \frac{\partial \psi_2}{\partial t}(T,\sigma)d\sigma\right). \tag{2.82}$$

Then, the regional average controllability problem on ω reduces to solving the equation:

$$\Lambda\left(\varphi_1,\varphi_0\right)=\chi_\omega^*\left(y_d^1,-y_d^2\right)-\mathcal{P}\left(\int_0^1\psi_1(T,\sigma)d\sigma,\int_0^1-\frac{\partial\psi_1}{\partial t}(T,\sigma)d\sigma\right), \tag{2.83}$$

and we have proved the following result

Theorem 2.7 *If the system (2.72) is weakly regionally averaged controllable on ω, then for any $(y_d^1,y_d^2)\in L^2(\omega)\times L^2(\omega)$ the equation (2.83) has a unique solution φ_0,φ_1, and the control*

$$u^*(t)=\left\langle f,\int_0^1\varphi(t,\sigma)d\sigma\right\rangle_{L^2(D)},$$

steers the average state of system (2.72) to (y_d^1,y_d^2) at time T in ω. Moreover, u^* solves the minimum problem (2.73).

Proof. We see that if the system (2.72) is weakly regionally averaged controllable on ω, then $\|.\|_G$ is a norm on the space G (Lemma 2.5).
We denote the completion of G with respect to the norm (2.75) again by G and G^* be its dual. We will show that (2.83) has a unique solution in G. For that purpose, multiply (2.74) by ψ_2 and integrate on Q to get

$$\int_Q\frac{\partial^2\varphi}{\partial t^2}(t,\sigma)\psi_2(t,\sigma)dxdt=\int_Q A^*(\sigma)\varphi(t,\sigma)\psi_2(t,\sigma)dxdt,$$

which gives

$$\begin{aligned}&\int_\Omega\left[\frac{\partial\varphi}{\partial t}(t,\sigma)\psi_2(t,\sigma)\right]_0^T dx-\int_\Omega\left[\varphi(t,\sigma)\frac{\partial\psi_2}{\partial t}(t,\sigma)\right]_0^T dx+\int_Q\varphi(t,\sigma)\frac{\partial^2\psi_2}{\partial t^2}(t,\sigma)dxdt\\ =&\int_Q A^*(\sigma)\varphi(t,\sigma)\psi_2(t,\sigma)dxdt.\end{aligned}$$

By the Green formula and with the boundary condition, we have

$$\langle\varphi_1,\psi_2(T,\sigma)\rangle-\left\langle\varphi_0,\frac{\partial\psi_2}{\partial t}(T,\sigma)\right\rangle=\int_0^T\left\langle f,\int_0^1\varphi(t,\sigma)d\sigma\right\rangle_{L^2(D)}\langle f,\varphi(t,\sigma)\rangle_{L^2(D)}dt,$$

then integrate on $(0,1)$ to obtain

$$\begin{aligned}\left\langle\varphi_1,\int_0^1\psi_2(T,\sigma)d\sigma\right\rangle-\left\langle\varphi_0,\int_0^1\frac{\partial\psi_2}{\partial t}(T,\sigma)d\sigma)\right\rangle &= \int_0^T\left|\left\langle f,\int_0^1\varphi(t,\sigma)d\sigma\right\rangle\right|_{L^2(D)}^2 dt\\ &= \|(\varphi_1,\varphi_0)\|_G^2.\end{aligned}$$

We have

$$\langle\Lambda\left(\varphi_1,\varphi_0\right),\left(\varphi_1,\varphi_0\right)\rangle=\int_0^T\left\langle f,\int_0^1\varphi(t,\sigma)d\sigma\right\rangle_{L^2(D)}^2 dt=\|(\varphi_1,\varphi_0)\|_G^2.$$

Hence, we conclude that Λ is an isomorphism between G and G^* [65]. Then (2.83) has a unique solution and

$$u^*(t) = \left\langle f, \int_0^1 \varphi(t,\sigma)d\sigma \right\rangle_{L^2(D)},$$

is a solution of the problem (2.73).

Now, let $u(t)$ and $v(t)$ are solutions of (2.73). We multiply (2.74) by $y_u(t,\sigma) - y_v(t,\sigma)$, apply Green formula and considering the boundary and initial conditions, we get

$$\begin{aligned}\int_0^T \langle f, \varphi(t,\sigma)\rangle_{L^2(D)} (u(t)-v(t))dt &= \left\langle \varphi_0, \left(\frac{\partial y_u}{\partial t}(T,\sigma) - \frac{\partial y_v}{\partial t}(T,\sigma)\right)\right\rangle_{L^2(D)} \\ &\quad - \langle \varphi_1, (y_u(T,\sigma) - y_v(T,\sigma))\rangle_{L^2(D)},\end{aligned}$$

integrate over $(0,1)$, we obtain

$$\begin{aligned}\int_0^T \left\langle f, \int_0^1 \varphi(t,\sigma)d\sigma \right\rangle_{L^2(D)} (u(t)-v(t))dt &= \left\langle \varphi_0, \int_0^1 \left(\frac{\partial y_u}{\partial t}(T,\sigma) - \frac{\partial y_v}{\partial t}(T,\sigma)\right) d\sigma\right\rangle_{L^2(D)} \\ &\quad - \left\langle \varphi_1, \int_0^1 (y_u(T,\sigma) - y_v(T,\sigma))d\sigma\right\rangle_{L^2(D)},\end{aligned}$$

then,

$$\int_0^T \left\langle f, \int_0^1 \varphi(t,\sigma)d\sigma \right\rangle_{L^2(D)} (u(t)-v(t))dt = 0.$$

Now, we have

$$\begin{aligned}\frac{1}{2}\mathcal{J}'(u^*(t))(u(t)-v(t)) &= \int_0^T (u^*(t))(u(t)-v(t))dt \\ &= \int_0^T \left\langle f, \int_0^1 \varphi(t,\sigma)d\sigma \right\rangle_{L^2(D)} (u(t)-v(t))dt.\end{aligned}$$

The unicity of u^* comes from the strict convexity of $\mathcal{J}$ and this $\mathcal{J}'(u^*)(u-v) = 0$ which establishes the optimality of u^*. ■

The case of internal pointwise actuator

In this case, we consider a hyperbolic system excited by controls which applied via internal pointwise actuators (b,δ) located at $b \in \Omega$

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = A(\sigma)y + \delta(x-b)u & \text{in } Q, \\ y(t,\xi) = 0 & \text{on } \Sigma, \\ y(0,x) = y_0(x), \dfrac{\partial y}{\partial t}(0,x) = y_1(x) & \text{in } \Omega. \end{cases} \tag{2.84}$$

Now, for a given $(\varphi_1, \varphi_0) \in G$, we consider the system (2.74) and define the mapping

$$\|(\varphi_1, \varphi_0)\|_G = \left(\int_0^T \left(\int_0^1 \varphi(t, b, \sigma) d\sigma \right)^2 dt \right)^{\frac{1}{2}}, \tag{2.85}$$

which is a norm on G if the system (2.84) is weakly regionally averaged controllable on ω, and we consider the following system

$$\begin{cases} \dfrac{\partial^2 \psi}{\partial t^2}(t, x, \sigma) = A(\sigma)\psi + \delta(x - b) \int_0^1 \varphi(t, b, \sigma) d\sigma & \text{in } Q, \\ \psi(0, x, \sigma) = y_0(x), \dfrac{\partial \psi}{\partial t}(0, x, \sigma) = y_1(x) & \text{in } \Omega, \\ \psi(t, \xi, \sigma) = 0 & \text{on } \Sigma, \end{cases} \tag{2.86}$$

which has a unique solution $\psi(t, x, \sigma)$ can be written as $\psi(t, x, \sigma) = \psi_1(t, x, \sigma) + \psi_2(t, x, \sigma)$, where $\psi_1(t, x, \sigma)$ is the solution of (2.78) and $\Psi_2(t, x, \sigma)$ is a solution of the following system

$$\begin{cases} \dfrac{\partial^2 \psi_2}{\partial t^2} = A(\sigma)\psi_2 + \delta(x - b) \int_0^1 \varphi(t, b, \sigma) d\sigma & \text{in } Q, \\ \psi_2(0, x) = 0, \dfrac{\partial \psi_2}{\partial t}(0, x) = 0 & \text{in } \Omega, \\ \psi_2(t, \xi) = 0 & \text{on } \Sigma. \end{cases} \tag{2.87}$$

For $(\varphi_1, \varphi_0) \in G$, let Λ be the operator defined by (2.81) where $\psi_2(t, x, \sigma)$ is the solution of (2.87). The regional averaged controllability problem on ω is then equivalent to the resolution of the equation:

$$\Lambda(\varphi_1, \varphi_0) = \chi_\omega^* \left(z_d^1, -z_d^2 \right) - \mathcal{P} \left(\int_0^1 \psi_1(T, \sigma) d\sigma, \int_0^1 -\frac{\partial \psi_1}{\partial t}(T, \sigma) d\sigma \right), \tag{2.88}$$

and we have proved the following result

Theorem 2.8 *If the system (2.84) is weakly regionally averaged controllable on ω, then for any $(y_d^1, y_d^2) \in L^2(\omega) \times L^2(\omega)$ the equation (2.88) has a unique solution φ_0, φ_1, and the control*

$$u^*(t) = \int_0^1 \varphi(t, b, \sigma) d\sigma,$$

steers the averaged state of system (2.84) to (y_d^1, y_d^2) at time T in ω. Moreover, u^* solves the minimum problem (2.73).

Proof. The proof of Theorem (2.8) is similar to that of Theorem (3.26). ■

CHAPTER 3

On the averaged no-regret control for parameter dependent systems with missing data

In this chapter, we study general and abstract control systems depending on a parameter and with missing data. By combining the low-regret technique of J.-L. Lions and the averaged control notion introduced recently by E. Zuazua, we prove that we can steer the averaged state of our system to a desired state using the notion of *averaged no-regret* and *averaged low-regret control*. More precisely, we prove the convergence of the averaged *low-regret control* to the *averaged no-regret control* and we prove that the problem we are considering has a unique averaged no-regret control that we characterize by a singular optimality system. As an example, we apply the described theory on parameter dependent electromagnetic wave equation with missing initial conditions.

3.1 Problem Statement

Let $\mathcal{V}$ be a Hilbert space of dual $\mathcal{V}'$, $\sigma \in (0,1)$ is an unknown parameter, $A(\sigma) \in L(\mathcal{V}, \mathcal{V}')$ a partial differential operator, U the Hilbert space of controls, and $B(\sigma) \in L(U, \mathcal{V}')$ a control operator. Let G be a nonempty closed vector subspace of the Hilbert space of missing data F, and $\beta \in L(F, \mathcal{V})$. The state equation related to the control $v \in U$ and to the missing data $g \in G$ is given by

$$A(\sigma)y = B(\sigma)v + \beta g. \tag{3.1}$$

Supposing that A is an isomorphism from $\mathcal{V}$ to $\mathcal{V}'$, (3.1) is well posed in $\mathcal{V}$. Denote by $y = y(v, g, \sigma)$ the unique solution to (3.1) depending on the control v, the missing data g and depends continuously

on σ. Also, we suppose that the operators $A(\sigma)$ and $B(\sigma)$ depend on σ continuously.
We want to choose a control v independently of σ and g in a way such that the averaged state function y approaches a given observation $y_d \in \mathcal{V}$, i.e. we want to minimize the quadratic cost functional

$$J(v,g) = \left\| \int_0^1 y(v,g,\sigma)\, d\sigma - y_d \right\|_{\mathcal{V}}^2 + N \|v\|_U^2, \tag{3.2}$$

where $N \in \mathbb{R}_+^*$.
We are concerned with optimal controls v of the $(3.1)-(3.2)$ with missing initial data g, i.e.

$$\inf_{v\in U} \mathcal{J}(v,g),\ \forall g \in G.$$

Since the subspace G is different from $\{0\}$, the above minimization problem has no sense (G having infinite elements). One idea is then to solve the minmax problem

$$\inf_{v\in U}\left(\sup_{g\in G}(\mathcal{J}(v,g))\right),$$

but $\mathcal{J}(v,g)$ is not upper bounded since $\sup\limits_{g\in G}(\mathcal{J}(v,g)) = +\infty$. A natural idea of Lions [40] is to search for controls v such that

$$\mathcal{J}(v,g) - \mathcal{J}(0,g) \le 0,\ \forall g \in G.$$

Those controls v are called *averaged no-regret controls.*

3.2 Averaged no-regret control & averaged low-regret control

First, we introduce the averaged no-regret control of (3.1)-(3.2). As in [57] and [19], it is defined by

Definition 3.1 *[20] We say that $u \in U$ is an averaged no-regret control for (3.1)-(3.2) if u is a solution to the following problem*

$$\inf_{v\in U}\left(\sup_{g\in G}(\mathcal{J}(v,g) - \mathcal{J}(0,g))\right), \tag{3.3}$$

we shall isolate g to a form where the classical theory of optimal control can be applied, the method is shown in the following Lemma

Lemma 3.1 *For all $v \in U$ and $g \in G$ we have*

$$\mathcal{J}(v,g) - \mathcal{J}(0,g) = \mathcal{J}(v,0) - \mathcal{J}(0,0) + 2\left\langle \beta^* \int_0^1 \zeta(v,\sigma)d\sigma, g \right\rangle_{G',G}, \tag{3.4}$$

where ζ is given by the following backward wave equation

$$A^*(\sigma)\zeta(v,\sigma) = \int_0^1 y(v,0,\sigma)d\sigma, \tag{3.5}$$

A^* (resp. β^*) being the adjoint of A (resp. β).

Proof. It's easy to check that for all $(v,g) \in U \times G$

$$\mathcal{J}(v,g) - \mathcal{J}(0,g) = \mathcal{J}(v,0) - \mathcal{J}(0,0) + 2\left(\int_0^1 y(v,0,\sigma)d\sigma, \int_0^1 y(0,g,\sigma)d\sigma \right)_{\mathcal{V}}.$$

Now, we introduce an adjoint state ζ defined by (3.5) and apply Green formula to get

$$\begin{aligned}
\left(\int_0^1 y(v,0,\sigma)d\sigma, \int_0^1 y(0,g,\sigma)d\sigma \right)_{\mathcal{V}} &= \int_0^1 \left(\int_0^1 y(v,0,\sigma)d\sigma, y(0,g,\sigma) \right)_{\mathcal{V}} d\sigma \\
&= \int_0^1 \langle A^*(\sigma)\zeta(v,\sigma), y(0,g,\sigma)\rangle_{\mathcal{V}',\mathcal{V}} d\sigma \\
&= \left\langle \beta^* \int_0^1 \zeta(v,\sigma)d\sigma, g \right\rangle_{G',G},
\end{aligned}$$

(notice that $A(\sigma)y(0,g,\sigma) = \beta g$). Then,

$$\mathcal{J}(v,g) - \mathcal{J}(0,g) = \mathcal{J}(v,0) - \mathcal{J}(0,0) + 2\left\langle \beta^* \int_0^1 \zeta(v,\sigma)d\sigma, g \right\rangle_{G',G}.$$

■

The averaged no-regret control seems to be difficult to characterize in this present form (see [55]), for this reason we relax the averaged no-regret control problem by making some quadratic perturbation. Thus, we define the averaged low-regret control notion.

Definition 3.2 *[19] We say that $u_\gamma \in U$ is an averaged low-regret control for (3.1)-(3.2) if u_γ is a solution to the following problem*

$$\inf_{v\in U}\left(\sup_{g\in G}(\mathcal{J}(v,g) - \mathcal{J}(0,g) - \gamma\|g\|_G^2) \right), \tag{3.6}$$

where γ is a strictly positive and small parameter.

Then according to (3.4), problem (3.6) is equivalent to

$$\inf_{v\in U}\left(\mathcal{J}\left(v,0\right)-\mathcal{J}\left(0,0\right)+\left(\sup_{g\in G}2\left\langle \beta^{*}\int_{0}^{1}\zeta(v,\sigma)d\sigma,g\right\rangle_{G',G}-\gamma\left\|g\right\|_{G}^{2}\right)\right).$$

which by means of Legendre–Fenchel transform (see [6] and [5]) is equivalent to the following problem

For any $\gamma>0$, find $u_{\gamma}\in U$ such that

$$\inf_{v\in U}\mathcal{J}_{\gamma}\left(v\right), \tag{3.7}$$

where

$$\mathcal{J}_{\gamma}\left(v\right)=\mathcal{J}\left(v,0\right)-\mathcal{J}\left(0,0\right)+\frac{1}{\gamma}\left\|\beta^{*}\int_{0}^{1}\zeta(v,\sigma)d\sigma\right\|_{G}^{2}. \tag{3.8}$$

3.3 Existence and characterization of the averaged low-regret control

In this section, our main objective would be to prove that the averaged low-regret problem (3.7)-(3.8) has a unique solution that converges to the unique solution of averaged no-regret control (3.3). In addition, we will give the equations that describe the low-regret control.

We begin by proving the existence of the averaged low-regret control for (3.1)-(3.2).

Theorem 3.1 *There exists a unique averaged low-regret control denoted by u_{γ} solution to the problem of minimization (3.7)-(3.8).*

Proof. From the definition of $\mathcal{J}_{\gamma}$, it's clear that for every $v\in U:\mathcal{J}_{\gamma}\left(v\right)\geq-\mathcal{J}\left(0,0\right)$, this means that (3.7)-(3.8) has a solution.

Let be then $(v_{n}^{\gamma})\in U$ a minimizing sequence (See Appendix) such that

$$\lim_{n\to\infty}\mathcal{J}_{\gamma}\left(v_{n}^{\gamma}\right)=d_{\gamma}.$$

We have,

$$\mathcal{J}_{\gamma}\left(v_{n}^{\gamma}\right)=\mathcal{J}\left(v_{n}^{\gamma},0\right)-\mathcal{J}\left(0,0\right)+\frac{1}{\gamma}\left\|\beta^{*}\int_{0}^{1}\zeta(v_{n}^{\gamma},\sigma)d\sigma\right\|_{G}^{2}\leq d_{\gamma}+1,$$

which implies that

$$\mathcal{J}\left(v_{n}^{\gamma},0\right)\leq C_{\gamma},$$

$$\frac{1}{\gamma}\left\|\beta^{*}\int_{0}^{1}\zeta(v_{n}^{\gamma},\sigma)d\sigma\right\|_{G}^{2}\leq C_{\gamma},$$

where C_γ is a positive constant independent of n. Hence, using the fact that

$$\mathcal{J}\left(v_n^\gamma, 0\right) = \left\| \int_0^1 y\left(v_n^\gamma, 0, \sigma\right) d\sigma - y_d \right\|_{\mathcal{V}}^2 + N \left\| v_n^\gamma \right\|_U^2 .$$

This implies the following estimates

$$\|v_n^\gamma\|_U \leq C_\gamma, \tag{3.9}$$

$$\left\| \int_0^1 y\left(v_n^\gamma, 0, \sigma\right) d\sigma \right\|_{\mathcal{V}} \leq C_\gamma, \tag{3.10}$$

$$\left\| \beta^* \int_0^1 \zeta(v_n^\gamma, \sigma) d\sigma \right\|_G \leq \sqrt{\gamma} C_\gamma, \tag{3.11}$$

From (3.9), we deduce that there exists a subsequence still denoted (v_n^γ) such that

$$v_n^\gamma \rightharpoonup u_\gamma \text{ weakly in } U.$$

Moreover, by continuity w.r.t. data and (3.10), we get

$$\|y\left(v_n^\gamma, 0, \sigma\right)\|_{\mathcal{V}} \leq C_\gamma, \tag{3.12}$$

then,

$$y\left(v_n^\gamma, 0, \sigma\right) \rightharpoonup y_\gamma \text{ weakly in } \mathcal{V}, \tag{3.13}$$

by passing to limit and uniqueness of limit we prove that $y_\gamma = y\left(u_\gamma, 0, \sigma\right)$. In view of (3.10),(3.13) and by the Lebesgue-dominated convergence theorem, we get

$$\int_0^1 y\left(v_n^\gamma, 0, \sigma\right) d\sigma \rightharpoonup \int_0^1 y\left(u_\gamma, 0, \sigma\right) d\sigma \text{ weakly in } \mathcal{V}.$$

Moreover, we have

$$A^*\left(\sigma\right) \zeta(v_n^\gamma, \sigma) = \int_0^1 y(v_n^\gamma, 0, \sigma) d\sigma \rightharpoonup \int_0^1 y\left(u_\gamma, 0, \sigma\right) d\sigma = A^*\left(\sigma\right) \zeta(u_\gamma, \sigma) \text{ weakly in } \mathcal{V},$$

as $A^*\left(\sigma\right)$ is an isomorphism, we have also

$$\zeta(v_n^\gamma, \sigma) \rightharpoonup \zeta(u_\gamma, \sigma) \text{ weakly in } \mathcal{V}.$$

In a manner similar to the convergence of $y(v_n^\gamma, 0, \sigma)$, we get

$$\int_0^1 \zeta(v_n^\gamma, \sigma) d\sigma \rightharpoonup \int_0^1 \zeta(u_\gamma, \sigma) d\sigma \text{ weakly in } \mathcal{V},$$

and according to the continuity of β^*, we deduce the following convergence

$$\beta^* \int_0^1 \zeta(v_n^\gamma, \sigma) d\sigma \rightharpoonup \beta^* \int_0^1 \zeta(u_\gamma, \sigma) d\sigma \text{ weakly in } \mathcal{V}.$$

This implies that

$$\begin{aligned}\mathcal{J}_\gamma(u_\gamma) &= \mathcal{J}(u_\gamma,0)-\mathcal{J}(0,0)+\frac{1}{\gamma}\left\|\beta^*\int_0^1\zeta(u_\gamma,\sigma)d\sigma\right\|_G^2\\ &\leq \inf_{n\in\mathbb{N}}\left(\mathcal{J}(v_n^\gamma,0)-\mathcal{J}(0,0)+\frac{1}{\gamma}\left\|\beta^*\int_0^1\zeta(v_n^\gamma,\sigma)d\sigma\right\|_G^2\right)\\ &\leq d_\gamma.\end{aligned}$$

Thus u_γ is a minimizing solution. Moreover, the uniqueness of u_γ follows from strict convexity and weak lower semi-continuity of the functional $\mathcal{J}_\gamma(v)$. ■

Now, we can now characterise the averaged low-regret control u_γ.

Theorem 3.2 *For all $\gamma>0$, the averaged low-regret control u_γ, solution of (3.7)-(3.8) is characterized by the unique solution $\{y_\gamma,\zeta_\gamma,\rho_\gamma,p_\gamma\}$ of the following optimality system*

$$\begin{cases} A(\sigma)y=B(\sigma)u,\\ A^*(\sigma)\zeta(u_\gamma,\sigma)=\int_0^1 y(u_\gamma,0,\sigma)d\sigma,\\ A(\sigma)\rho_\gamma=\dfrac{1}{\gamma}\beta\beta^*\int_0^1\zeta(u_\gamma,\sigma)d\sigma,\\ A^*(\sigma)p_\gamma=\int_0^1\big(\rho_\gamma+y(u_\gamma,0,\sigma)\big)d\sigma-y_d,\\ \int_0^1 B^*p_\gamma+Nu_\gamma=0 \text{ in } U.\end{cases}\tag{3.14}$$

Proof. A first order necessary condition gives for every $w\in L^2(\Sigma_0)$

$$\begin{aligned}(\textstyle\int_0^1 y(u_\gamma,0,\sigma)d\sigma-y_d,\int_0^1 y(w,0,\sigma)d\sigma)_{\mathcal{V}}+N(u_\gamma,w)_U+\\ \frac{1}{\gamma}(\beta^*\textstyle\int_0^1\zeta(u_\gamma,\sigma)d\sigma,\beta^*\int_0^1\zeta(w,\sigma)d\sigma)_G\geq 0.\end{aligned}\tag{3.15}$$

Let $\rho_\gamma=\rho(u_\gamma,0)$ be the solution of

$$A(\sigma)\rho_\gamma=\frac{1}{\gamma}\beta\beta^*\int_0^1\zeta(u_\gamma,\sigma)d\sigma,\tag{3.16}$$

Then,

$$\begin{aligned}\frac{1}{\gamma}(\beta^*\int_0^1\zeta(u_\gamma,\sigma)d\sigma,\beta^*\int_0^1\zeta(w,\sigma)d\sigma)_G &= (\int_0^1 A(\sigma)\rho_\gamma d\sigma,\int_0^1\zeta(w,\sigma)d\sigma)_{\mathcal{V}}\\ &= \int_0^1(\rho_\gamma,\int_0^1 y(w,0,\sigma)d\sigma)_{\mathcal{V}}d\sigma\\ &= (\int_0^1\rho_\gamma d\sigma,\int_0^1 y(w,0,\sigma)d\sigma)_{\mathcal{V}}.\end{aligned}$$

Hence, the optimality condition (3.15) is equivalent to

$$\left(\int_0^1\big(y(u_\gamma,0,\sigma)+\rho_\gamma\big)d\sigma-y_d,\int_0^1 y(w,0,\sigma)d\sigma\right)_{\mathcal{V}}+N(u_\gamma,w)_U\geq 0.\tag{3.17}$$

Again, let's construct an adjoint state $p_\gamma = p(u_\gamma)$ solution to

$$A^*(\sigma)\, p_\gamma = \int_0^1 \big(\rho_\gamma + y(u_\gamma, 0, \sigma)\big)\, d\sigma - y_d. \tag{3.18}$$

Then (3.15) is equivalent to

$$\left(\int_0^1 B^*(\sigma)\, p_\gamma d\sigma + N u_\gamma, w\right)_U \geq 0 \;\forall w \in U. \tag{3.19}$$

Since U is a Hilbert space, we also have

$$\left(\int_0^1 B^*(\sigma)\, p_\gamma d\sigma + N u_\gamma, w\right)_U \leq 0 \;\forall w \in U, \tag{3.20}$$

which gives (3.14). This ends the proof. ■

3.4 Existence and characterization of the averaged no-regret control

Now we give a singular optimality system for the approximate averaded no-regret control.

Theorem 3.3 *When $\gamma \to 0$, the averaged low-regret control u_γ converges to the averaged no-regret control u, solution of (3.3).*

Proof. As u_γ is an averaged low-regret control, then for every $v \in U$

$$\mathcal{J}(u_\gamma, 0) - \mathcal{J}(0,0) + \frac{1}{\gamma}\left\|\beta^* \int_0^1 \zeta(u_\gamma, \sigma) d\sigma\right\|_G^2 \leq \mathcal{J}(v, 0) - \mathcal{J}(0,0) + \frac{1}{\gamma}\left\|\beta^* \int_0^1 \zeta(v, \sigma) d\sigma\right\|_G^2,$$

take $v = 0$ to find

$$\left\|\int_0^1 y(u_\gamma, 0, \sigma)\, d\sigma - y_d\right\|_{\mathcal{V}}^2 + N\|u_\gamma\|_U^2 + \frac{1}{\gamma}\left\|\beta^* \int_0^1 \zeta(u_\gamma, \sigma) d\sigma\right\|_G^2 \leq \mathcal{J}(0,0),$$

from which, we deduce the following bounds

$$\|u_\gamma\|_U \leq C, \tag{3.21}$$

$$\left\|\int_0^1 y(u_\gamma, 0, \sigma)\, d\sigma\right\|_{\mathcal{V}} \leq C, \tag{3.22}$$

$$\left\|\beta^* \int_0^1 \zeta(u_\gamma, \sigma) d\sigma\right\|_G^2 \leq \gamma C, \tag{3.23}$$

where C is a positive constant independent of σ, then, by (3.21), we find that there exists a subsequence still denoted u_γ such that

$$u_\gamma \rightharpoonup u \text{ weakly in } U.$$

It remains to prove that u is an averaged no-regret control i.e. a solution for (3.3). It's clear that for all $v \in U$

$$\mathcal{J}(u_\gamma, g) - \mathcal{J}(0, g) - \gamma \|g\|_G^2 \leq \mathcal{J}(v, g) - \mathcal{J}(0, g),$$

then,

$$\mathcal{J}(u_\gamma, g) - \mathcal{J}(0, g) - \gamma \|g\|_G^2 \leq \sup_{g \in G}(\mathcal{J}(v, g) - \mathcal{J}(0, g)),$$

make $\gamma \to 0$ to get

$$\mathcal{J}(u, g) - \mathcal{J}(0, g) \leq \sup_{g \in G}(\mathcal{J}(v, g) - \mathcal{J}(0, g)),$$

i.e. u is an averaged no-regret control. ■

Finally, we can present the following theorem giving a full characterization the averaged no-regret control.

Theorem 3.4 *The averaged no-regret control u is characterized by the following optimality system*

$$\begin{cases} A(\sigma)y = B(\sigma)u, \\ A^*(\sigma)\zeta(u,\sigma) = \int_0^1 y(u,0,\sigma)d\sigma, \\ A(\sigma)\rho = \lambda, \\ A^*(\sigma)p = \int_0^1 (\rho + y(u,0,\sigma))\, d\sigma - y_d, \\ \int_0^1 B^* p + Nu = 0 \ \text{in } U, \end{cases} \tag{3.24}$$

where $\lambda(x) = \lim_{\gamma \to 0} \frac{1}{\gamma} \beta\beta^* \int_0^1 \zeta(u,\sigma)d\sigma$ *weakly in* U.

Proof. From Theorem 3.3, we know that

$$u_\gamma \rightharpoonup u \text{ weakly in } U,$$

then, as $B(\sigma)$ is bounded, we find

$$B(\sigma)u_\gamma \rightharpoonup B(\sigma)u \text{ weakly in } \mathcal{V}.$$

Also, by continuity w.r.t. data $y(u_\gamma, 0, \sigma)$ converges weakly to $y(u, 0, \sigma)$ in $\mathcal{V}$, and from the continuity of $A(\sigma)$, we deduce that

$$A(\sigma)y(u_\gamma, 0, \sigma) \rightharpoonup A(\sigma)y(u, 0, \sigma) \text{ weakly in } \mathcal{V},$$

from the limit uniqueness, we deduce

$$A(\sigma)y(u,0,\sigma)=B(\sigma)u.$$

By reasoning by contradiction and from (3.23), we have

$$\left\|\beta^*\int_0^1\zeta(u_\gamma,\sigma)d\sigma\right\|_G^2\leq\gamma C\Rightarrow\left\|\beta^*\int_0^1\zeta(u_\gamma,\sigma)d\sigma\right\|_G\leq\gamma C,$$

i.e. $\beta^*\int_0^1\zeta(u_\gamma,\sigma)d\sigma$ is bounded in $\mathcal{V}$, then as β is bounded, $\dfrac{1}{\gamma}\beta\beta^*\int_0^1\zeta(u_\gamma,\sigma)d\sigma$ is also bounded, then, we get

$$\frac{1}{\gamma}\beta\beta^*\int_0^1\zeta(u_\gamma,\sigma)d\sigma\rightharpoonup\lambda\text{ weakly in }\mathcal{V},$$

Likewise, $A(\sigma)\rho_\gamma$ is bounded and by isomorphism of $A(\sigma)$, we know that ρ_γ is bounded also and converges to ρ, then

$$A(\sigma)\rho_\gamma\rightharpoonup A(\sigma)\rho\text{ weakly in }\mathcal{V},$$

hence,

$$A(\sigma)\rho=\lambda.$$

Moreover, from the boundness of y_γ and ρ_γ, we deduce the boundness of $A(\sigma)p_\gamma$ and therefore p_γ is bounded in $\mathcal{V}$, and we get

$$A^*(\sigma)\,p=\int_0^1(\rho+y(u,0,\sigma))\,d\sigma-y_d.$$

At last, pass to limit in the variational inequality of (3.14) and use weak convergences of u_γ,p_γ to u,p resp. to get

$$\int_0^1 B^*p+Nu=0\text{ in }U.$$

■

3.5 Application (Optimal control of parameter dependent electromagnetic wave equation with missing initial conditions)

In this section, we apply the above method throughout the example given below in situation boundary control and missing initial conditions.

Let an open bounded subset Ω of $\mathbb{R}^n$ with a regular boundary Γ and $T>0$. We set $Q=\Omega\times]0,T[$, $\Sigma=\Gamma\times]0,T[$. We consider the following linear electromagnetic wave equation depending on an unknown parameter σ with a boundry control action

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}-\Delta y+p\left(x,\sigma\right)y=0 & \text{in } Q,\\ y=\begin{cases} v & \text{on } \Sigma_0,\\ 0 & \text{on } \Sigma\setminus\Sigma_0,\end{cases} \\ y\left(x,0\right)=y_0\left(x\right);\dfrac{\partial y}{\partial t}\left(x,0\right)=y_1\left(x\right) & \text{in } \Omega,\end{cases} \tag{3.25}$$

where $p\in L^\infty\left(\Omega\right)$ is the potential term supposed dependent on an unknown parameter $\sigma\in(0,1)$ presents the dielectric permittivity, v presents a boundary control action in $L^2\left(\Sigma_0\right)$, $y_0\in H_0^1\left(\Omega\right)$, $y_1\in L^2\left(\Omega\right)$ are the initial position and velocity respectively, both supposed unknown and all independent of the parameter σ. The wave equation (3.25) has a unique solution $y\left(v,y_0,y_1,\sigma\right)$ in $C([0,T];H^1(\Omega))\cap C^1([0,T];L^2(\Omega))$ [29]. We denote $g=\left(y_0,y_1\right)\in G=H_0^1\left(\Omega\right)\times L^2\left(\Omega\right)$ the initial missing data.

We want to choose a control v independently of σ and g in a way such that the average state function y approaches a given observation $y_d\in L^2\left(Q\right)$, i.e. we want to minimize the quadratic cost functional

$$\mathcal{J}\left(v,g\right)=\left\|\int_0^1 y\left(v,g,\sigma\right)d\sigma-y_d\right\|^2_{L^2(Q)}+N\left\|v\right\|^2_{L^2(\Sigma_0)}, \tag{3.26}$$

where $y_d\in L^2\left(Q\right)$ is a given observation and $N\in\mathbb{R}_+^*$.

Lemma 3.2 *For all $v\in L^2\left(\Sigma_0\right)$ and $g\in G$ we have*

$$\begin{aligned}\mathcal{J}\left(v,g\right)-\mathcal{J}\left(0,g\right) &= \mathcal{J}\left(v,0\right)-\mathcal{J}\left(0,0\right)\\ &\quad -2\int_\Omega y_0\left(x\right)\int_0^1\frac{\partial\zeta}{\partial t}\left(x,0\right)d\sigma dx+2\int_\Omega y_1\left(x\right)\int_0^1\zeta\left(x,0\right)d\sigma dx,\end{aligned} \tag{3.27}$$

where ζ is given by the following backward wave equation

$$\begin{cases} \dfrac{\partial^2 \zeta}{\partial t^2}-\Delta\zeta+p\left(x,\sigma\right)\zeta=\int_0^1 y\left(v,0,\sigma\right)d\sigma & \text{in } Q,\\ \zeta=0 & \text{on } \Sigma,\\ \zeta\left(x,T\right)=0,\dfrac{\partial\zeta}{\partial t}\left(x,T\right)=0 & \text{in } \Omega,\end{cases} \tag{3.28}$$

which has a unique solution in $C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$ [29].

Proof. It's easy to check that for all $(v,g)\in L^2\left(\Sigma_0\right)\times G$

$$\mathcal{J}(v,g)-\mathcal{J}(0,g)=\mathcal{J}(v,0)-\mathcal{J}(0,0)+2\int_Q\left(\int_0^1 y(v,0)d\sigma\right)\left(\int_0^1 y(0,g)d\sigma\right)dxdt.$$

Now,we introduce an adjoint state ζ defined by (3.28) and apply Green formula to get

$$\begin{aligned}\int_Q \left(\int_0^1 y(v,0)d\sigma\right)\left(\int_0^1 y(0,g)d\sigma\right) dxdt &= \int_0^T\int_\Omega \left(\frac{\partial^2\zeta}{\partial t^2} - \Delta\zeta + p\left(x,\sigma\right)\zeta\right)\left(\int_0^1 y(0,g)d\sigma\right) dxdt \\ &= -2\int_\Omega y_0\left(x\right)\int_0^1 \frac{\partial\zeta}{\partial t}\left(v,0\right)d\sigma dx + 2\int_\Omega y_1\left(x\right)\int_0^1 \zeta\left(v,0\right)d\sigma dx.\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{J}\left(v,g\right) - \mathcal{J}\left(0,g\right) &= \mathcal{J}\left(v,0\right) - \mathcal{J}\left(0,0\right) \\ &\quad -2\int_\Omega y_0\left(x\right)\int_0^1 \frac{\partial\zeta}{\partial t}\left(x,0\right)d\sigma dx + 2\int_\Omega y_1\left(x\right)\int_0^1 \zeta\left(x,0\right)d\sigma dx.\end{aligned}$$

■

From section 3.2, the averaged low-regret control associated with (3.25)-(3.26) is defined by

$$\inf_{v\in L^2(\Sigma_0)} \mathcal{J}_\gamma\left(v\right), \tag{3.29}$$

where

$$\mathcal{J}_\gamma\left(v\right) = \mathcal{J}\left(v,0\right) - \mathcal{J}\left(0,0\right) + \frac{1}{\gamma}\left\|\int_0^1 \frac{\partial\zeta\left(v,\sigma\right)}{\partial t}\left(x,0\right)d\sigma\right\|^2_{H_0^1(\Omega)} + \frac{1}{\gamma}\left\|\int_0^1 \zeta\left(v,\sigma\right)\left(x,0\right)d\sigma\right\|^2_{L^2(\Omega)}. \tag{3.30}$$

Now, we can now characterise the unique averaged low-regret control u_γ .

Theorem 3.5 *For all $\gamma > 0$, the averaged low-regret control u_γ, solution of (3.29)-(3.30) is characterized by the unique solution $\left\{y_\gamma, \zeta_\gamma, \rho_\gamma, p_\gamma\right\}$ of the following optimality system*

$$\begin{cases} \dfrac{\partial^2 y_\gamma}{\partial t^2} - \Delta y_\gamma + p(x,\sigma) y_\gamma = 0, \\ \dfrac{\partial^2 \zeta_\gamma}{\partial t^2} - \Delta \zeta_\gamma + p(x,\sigma)\zeta_\gamma = \int_0^1 y\left(u_\gamma, 0, \sigma\right) d\sigma, \\ \dfrac{\partial^2 \rho_\gamma}{\partial t^2} - \Delta \rho_\gamma + p(x,\sigma)\rho_\gamma = 0, \\ \dfrac{\partial^2 q_\gamma}{\partial t^2} - \Delta q_\gamma + p(x,\sigma) q_\gamma = \int_0^1 \left(\rho_\gamma + y\left(u_\gamma, 0, \sigma\right)\right) d\sigma - y_d & \textit{in } Q, \\ y_\gamma = \begin{cases} u_\gamma & \textit{on } \Sigma_0 \\ 0 & \textit{on } \Sigma \setminus \Sigma_0 \end{cases}, \zeta_\gamma = 0, \\ \rho_\gamma = 0, q_\gamma = 0 & \textit{on } \Sigma, \\ y_\gamma(0,x) = 0, \dfrac{\partial y_\gamma}{\partial t}(0,x) = 0, \\ \zeta_\gamma(x,T) = 0, \dfrac{\partial \zeta_\gamma}{\partial t}(x,T) = 0, \\ \rho_\gamma\left(x,0\right) = -\dfrac{1}{\gamma}\int_0^1 \dfrac{\partial \zeta\left(u_\gamma\right)}{\partial t}\left(x,0\right) d\sigma, \dfrac{\partial \rho_\gamma}{\partial t}\left(x,0\right) = \dfrac{1}{\gamma}\int_0^1 \zeta\left(u_\gamma\right)\left(x,0\right) d\sigma, \\ q_\gamma(T,x) = 0, \dfrac{\partial q_\gamma}{\partial t}(T,x) = 0 & \textit{in } \Omega, \end{cases} \tag{3.31}$$

with

$$u_\gamma = \frac{1}{N}\int_0^1 \frac{\partial q_\gamma}{\partial \eta} d\sigma \ \textit{ in } L^2\left(\Sigma_0\right). \tag{3.32}$$

Proof. A first order necessary condition gives for every $w \in L^2\left(\Sigma_0\right)$

$$\begin{aligned} &(\textstyle\int_0^1 y(u_\gamma,0)d\sigma - y_d, \int_0^1 y(w,0)d\sigma)_{L^2(Q)} + N(u_\gamma, w)_{L^2(\Sigma_0)} + \\ &(\tfrac{1}{\gamma}\textstyle\int_0^1 \dfrac{\partial \zeta\left(u_\gamma\right)}{\partial t}(0)d\sigma, \int_0^1 \dfrac{\partial \zeta\left(w\right)}{\partial t}(0)d\sigma)_{L^2(\Omega)} + (\tfrac{1}{\gamma}\textstyle\int_0^1 \zeta\left(u_\gamma\right)(0)d\sigma, \int_0^1 \zeta\left(w\right)(0)d\sigma)_{L^2(\Omega)} = 0. \end{aligned} \tag{3.33}$$

Let $\rho_\gamma = \rho(u_\gamma, 0)$ be the solution of

$$\begin{cases} \dfrac{\partial^2 \rho_\gamma}{\partial t^2} - \Delta \rho_\gamma + p\left(x,\sigma\right)\rho_\gamma = 0 & \text{in } Q, \\ \rho_\gamma = 0 & \text{on } \Sigma, \\ \rho_\gamma\left(x,0\right) = -\dfrac{1}{\gamma}\int_0^1 \dfrac{\partial \zeta\left(u_\gamma\right)}{\partial t}\left(x,0\right) d\sigma; \dfrac{\partial \rho_\gamma}{\partial t}\left(x,0\right) = \dfrac{1}{\gamma}\int_0^1 \zeta\left(u_\gamma\right)\left(x,0\right) d\sigma & \text{in } \Omega. \end{cases} \tag{3.34}$$

Hence, the optimality condition (3.33) is equivalent to

$$\left(\int_0^1 y(u_\gamma,0) + \rho_\gamma d\sigma - y_d, \int_0^1 y(w,0)d\sigma - y\left(0,0\right)\right)_{L^2(Q)} + N(u_\gamma, w)_{L^2(\Sigma_0)} = 0. \tag{3.35}$$

Again, let's construct an adjoint state $q_\gamma = q(u_\gamma)$ solution to

$$\begin{cases} \dfrac{\partial^2 q_\gamma}{\partial t^2} - \Delta q_\gamma + p\left(x,\sigma\right) q_\gamma = \int_0^1 \left(\rho_\gamma + y\left(u_\gamma, 0, \sigma\right)\right) d\sigma - y_d & \text{in } Q, \\ q_\gamma = 0 & \text{on } \Sigma, \\ q_\gamma\left(x,T\right) = 0, \dfrac{\partial q_\gamma}{\partial t}\left(x,T\right) = 0 & \text{in } \Omega. \end{cases} \tag{3.36}$$

Then (3.33) is equivalent to

$$u_\gamma = \frac{1}{N}\int_0^1 \frac{\partial q_\gamma}{\partial \eta} d\sigma \text{ in } L^2(\Sigma_0). \tag{3.37}$$

which gives (3.32). This ends the proof. ■

Now we give a singular optimality system for the approximate no-regret control. Before doing this we give some a priori estimates as follows

Proposition 3.1 *There is some $C > 0$ independent of γ such that*

$$\|u_\gamma\|_{L^2(\Sigma_0)} \leq C, \tag{3.38}$$

$$\left\|\int_0^1 y(u_\gamma, 0, \sigma)\, d\sigma\right\|_{L^2(Q)} \leq C, \tag{3.39}$$

$$\|y(u_\gamma, 0, \sigma)\|_{L^2(Q)} \leq C, \tag{3.40}$$

$$\left\|\int_0^1 \frac{\partial \zeta(u_\gamma, \sigma)}{\partial t}(x, 0)\, d\sigma\right\|_{H^{-1}(\Omega)} \leq C\sqrt{\gamma}, \tag{3.41}$$

$$\left\|\int_0^1 \zeta(u_\gamma, \sigma)(x, 0)\, d\sigma\right\|_{L^2(\Omega)} \leq C\sqrt{\gamma}, \tag{3.42}$$

$$\|\rho_\gamma\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C, \tag{3.43}$$

$$\|q_\gamma\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C, \tag{3.44}$$

Proof. u_γ solves the optimal problem (3.29)(3.30), and we have particularly

$$\mathcal{J}_\gamma(u_\gamma) \leq \mathcal{J}_\gamma(0).$$

Then, from the definition of $\mathcal{J}$

$$\mathcal{J}(u_\gamma, 0) + \frac{1}{\gamma}\left\|\int_0^1 \frac{\partial \zeta(u_\gamma, \sigma)}{\partial t}(x, 0)\, d\sigma\right\|^2_{L^2(\Omega)} + \frac{1}{\gamma}\left\|\int_0^1 \zeta(u_\gamma, \sigma)(x, 0)\, d\sigma\right\|^2_{L^2(\Omega)} \leq \mathcal{J}(0, 0),$$

this gives (3.38),(3.39), (3.41) and (3.42). The bound (3.40) follows by a way similar to (3.12). From energy conservation property with (3.41) and (3.42).

$$E_{\rho_\gamma}(t) = \frac{1}{2}\int_\Omega \left[\left|\frac{\partial \rho_\gamma}{\partial t}\right|^2 + |\nabla \rho_\gamma|^2 + q(x, \sigma)|\rho_\gamma|^2\right] dx = E_{\rho_\gamma}(0) \leq C,$$

we find (3.43).

To get q_γ estimates, just reverse the time variable by taking $s = T - t$ to find (3.44). ■

Finally, we can present the following theorem giving a full characterization the average no-regret control.

Theorem 3.6 *The average no-regret control u is characterized by the following optimality system*

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \Delta y + p(x,\sigma) y = 0, \\ \dfrac{\partial^2 \zeta}{\partial t^2} - \Delta \zeta + p(x,\sigma)\zeta = \int_0^1 y(u,0,\sigma)\, d\sigma, \\ \dfrac{\partial^2 \rho}{\partial t^2} - \Delta \rho + p(x,\sigma)\rho = 0, \\ \dfrac{\partial^2 q}{\partial t^2} - \Delta q + p(x,\sigma) q = \int_0^1 (\rho + y(u,0,\sigma))\, d\sigma - y_d & \text{in } Q, \\ y = \begin{cases} u & \text{on } \Sigma_0 \\ 0 & \text{on } \Sigma \setminus \Sigma_0 \end{cases}, \zeta = 0 \\ \rho = 0; q = 0 & \text{on } \Sigma, \\ y(0,x) = 0, \dfrac{\partial y}{\partial t}(0,x) = 0, \\ \zeta(x,T) = 0, \dfrac{\partial \zeta}{\partial t}(x,T) = 0, \\ \rho(x,0) = \lambda_1(x), \dfrac{\partial \rho}{\partial t}(x,0) = \lambda_2(x), \\ q(T,x) = 0, \dfrac{\partial q}{\partial t}(T,x) = 0 & \text{in } \Omega, \end{cases} \tag{3.45}$$

with

$$u = \frac{1}{N}\int_0^1 \frac{\partial p}{\partial \eta} d\sigma \text{ in } L^2(\Sigma_0),$$

and

$\lambda_1(x) = \lim_{\gamma\to 0} -\dfrac{1}{\gamma}\int_0^1 \dfrac{\partial \zeta(u_\gamma)}{\partial t}(x,0)\, d\sigma$ *weakly in* $H_0^1(\Omega)$,

$\lambda_2(x) = \lim_{\gamma\to 0} \dfrac{1}{\gamma}\int_0^1 \zeta(u_\gamma)(x,0)\, d\sigma$ *weakly in* $L^2(\Omega)$.

Proof. From (3.40) continuity w.r.t data, we can deduce that

$$y(u_\gamma,0,\sigma) \rightharpoonup y(u,0,\sigma) \text{ weakly in } L^2(\Omega),$$

solution to

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \Delta y + p(x,\sigma) y = 0 & \text{in } Q, \\ y = \begin{cases} u & \text{on } \Sigma_0, \\ 0 & \text{on } \Sigma \setminus \Sigma_0, \end{cases} \\ y(x,0) = 0; \dfrac{\partial y}{\partial t}(x,0) = 0 & \text{in } \Omega. \end{cases}$$

Again, by (3.39) and dominated convergence theorem

$$\int_0^1 y(u_\gamma,0,\sigma)\, d\sigma \rightharpoonup \int_0^1 y(u,0,\sigma)\, d\sigma \text{ weakly in } L^2(\Sigma_0).$$

The rest of equations in (3.45) leads by a similar way, except the convergences of initial data $\rho(x,0)$, $\dfrac{\partial \rho}{\partial t}(x,0)$ which will be as follows.

From (3.41) and (3.42) we deduce the convergences of

$$-\frac{1}{\gamma}\frac{\partial\zeta\left(u_{\gamma},\sigma\right)}{\partial t}\left(x,0\right)\rightharpoonup\lambda_{1}\left(x\right)\ \text{ weakly in }H_{0}^{1}\left(\Omega\right),$$

and

$$\frac{1}{\gamma}\zeta\left(u_{\gamma},\sigma\right)\left(x,0\right)\rightharpoonup\lambda_{2}\left(x\right)\ \text{ weakly in }L^{2}\left(\Omega\right).$$

■

Conclusion and future perspectives

Main Contributions

In this thesis, we have studied the controllability and the optimal control of some distributed systems with missing data and depending on an unknown parameter.

In Chapter 2, we have proved the averaged null controllability for a wave equation with an unknown velocity of propagation parameter under the effect of boundary control. We have defined the averaged energy where we have proved its conservation. Next, we have established a fundamental theorem containing an averaged inverse inequality which is the key point to prove the averaged null controllability of the wave equation for a large enough time. Then we have applied the famous Hilbert uniqueness method which was introduced by Lions to construct an independent parameter control that steers the averaged state of a wave equation containing an unknown to zero. In the same way, we have treated the problem of the vibrating plate equation. Afterward, we have extended the usual results of regional controllability to hyperbolic parameter dependent systems. We have created a new concept which is regional averaged controllability and we have used an approach based on regional controllability tools in connection with averaged control structure based on an extension of the Hilbert uniqueness method devoted to the calculation of the control that steers the state average (with respect to such a parameter) towards the desired state only on a given part of the system evolution domain.

In Chapter 3, , we have studied general and abstract control systems depending on a parameter and with missing data. we have given a characterization (optimality system) using the averaged no-regret control method. more precisely, we have proved the convergence of the averaged low-regret control to the averaged no-regret control for which we have obtained a singular optimality system.

Hence, we have proved that we can bring the averaged state of our system to a desired state. Then , we apply the described theory on parameter dependent electromagnetic wave equation with missing initial conditions.

Perspectives

Hereafter, we list some possible developments for future research works.

- Concerning averaged regional controllability introduced in Chapter 2, to further generalize the previous treatise, one should consider the following case
  - the control operator B is also dependent on the parameter σ.
  - The case of boundary subregion.
  - When the systems considered are parabolic parameter dependent systems or fractional order systems.
- Another perspective that we could be interested in generalizing the result obtained in chapter 3 to the case of other systems which have many biomedical applications.

Appendix

Definition 1 [59] A family $\{S(t)\}_{t\geq 0}$ of operators in $\mathcal{L}(\mathcal{V})$ (where $\mathcal{V}$ is the state space) is a strongly continuous semi-group or C_0-semi-group on $\mathcal{V}$ if

i) $S(0) = I$.

ii) $S(t+s) = S(t)S(s)$, for all $s, t \geq 0$.

iii) $\lim\limits_{t\to 0^+} S(t)x = x \ \ \forall x \in \mathcal{V}$.

Definition 2 [59] The linear operator $A : D(A) \to \mathcal{V}$, defined by

$$D(A) = \left\{ x \in \mathcal{V} \ : \ \lim_{t\to 0} \frac{S(t)x - x}{t} \text{ existe} \right\}.$$

$$Ax = \lim_{t\to 0} \frac{S(t)x - x}{t}, \ \forall x \in D(A).$$

is called the infinitesimal generator of the semi-group S.

Definition 3 [59] Let $\Delta = \{z \in \mathbb{C} : \varphi_1 \leq \arg z \leq \varphi_2, \varphi_1 \leq 0 \leq \varphi_2\}$ a sector in $\mathbb{C}$. A family $\{S(z)\}_{z\in\Delta}$ of linear operators bounded on H is said to be an analytical (holomorphic) semi-group in Δ if it satisfies the following conditions:

i) $S(z_1 + z_2) = S(z_1)S(z_2), \forall \ z_1, z_2 \in \Delta$.

ii) $S(0) = I$.

iii) $\lim\limits_{z\to 0, z\in\Delta} S(z)x = x \ \forall x \in H$.

iv) The application $z \in \Delta^* = \Delta \backslash \{0\} \mapsto S(z)\, x \in H$ is analytical, $\forall x \in H$.

Proprety 1 [7] The adjoint A^* of A generates the semi-group $\{S^*(t)\}_{t\geq 0}$ adjoint of $\{S(t)\}_{t\geq 0}$ which is also strongly continuous on the dual $\mathcal{V}'$of $\mathcal{V}$.

Definition 4 Let $\mathcal{J} : U \subset X \to Y$ be an operator with Banach spaces X, Y and $U \neq \emptyset$ open. $\mathcal{J}$

is called directionally differentiable at $x \in U$ if the limit

$$D\mathcal{J}(x,h) = \lim_{t \to 0^+} \frac{\mathcal{J}(x+th) - \mathcal{J}(x)}{t} \in Y,$$

exists for all $h \in X$. $\mathcal{J}$ is called Gâteaux differentiable at $x \in U$ if $\mathcal{J}$ is directionally differentiable at x and the directional derivative $\mathcal{J}\prime(x) : h \in X \to D\mathcal{J}(x,h) \in Y$ is bounded and linear,i.e., $\mathcal{J}(x) \in \mathcal{L}(X,Y)$.

Theorem 1 [60] **(Integration by Parts Formulas)** Let $\Omega \subset \mathbb{R}^n$, be a C^1- domain. For vector fields

$$\mathbf{V} = (V_1, V_2, \ldots, V_n) : \Omega \to \mathbb{R}^n,$$

with $\mathbf{V} \in C^1\left(\overline{\Omega}\right)$, the **Gauss divergence formula** holds

$$\int_\Omega \operatorname{div} \mathbf{V} dx = \int_\Gamma \mathbf{V}.\eta d\mu, \tag{1}$$

where $\operatorname{div} \mathbf{V} = \sum_{i=0}^{n} \frac{\partial V_j}{\partial x_i}$, η denotes the outward normal unit vector to Γ, and $d\mu$ is the "surface" measure on Γ.

A number of useful identities can be derived from (1). Applying (1) to $f\mathbf{V}$,with $f \in C^1\left(\overline{\Omega}\right)$, and recalling the identity

$$\operatorname{div}(f\mathbf{V}) = f.\operatorname{div}\mathbf{V} + \nabla f.\mathbf{V},$$

we obtain the following integration by parts formula

$$\int_\Omega f \operatorname{div} \mathbf{V} dx = \int_\Gamma f\mathbf{V}.\eta d\mu + \int_\Omega \nabla f.\mathbf{V} dx. \tag{2}$$

Choosing $\mathbf{V} = \nabla g, g \in C^2(\Omega) \cap \in C^1\left(\overline{\Omega}\right)$, since $\operatorname{div} \nabla g = \triangle g$ and $\triangle g.\eta = \frac{\partial g}{\partial \eta}$, the following Green's identity follows

$$\int_\Omega f \triangle g dx = \int_\Gamma f \frac{\partial g}{\partial \eta} d\mu - \int_\Omega \nabla f \nabla g dx. \tag{3}$$

In particular, the choice $f = 1$ yields

$$\int_\Omega f \triangle g dx = \int_\Gamma \frac{\partial g}{\partial \eta} d\mu. \tag{4}$$

If also $f \in C^2(\Omega) \cap \in C^1\left(\overline{\Omega}\right)$ interchanging the roles of g and f in (3) and subtracting, we derive a second Green's identity

$$\int_\Omega f \triangle g dx - \int_\Omega \triangle f g dx = \int_\Gamma \left(f \frac{\partial g}{\partial \eta} - g \frac{\partial f}{\partial \eta} \right) d\mu. \tag{5}$$

Theorem 2 Let any real numbers a , b and let p, q be real numbers connected by the relationship $\frac{1}{p} + \frac{1}{q} = 1$.

Then we have the Cauchy inequality

$$ab \leq \frac{1}{2}\left(a^2 + b^2\right).$$

Theorem 3 [45] Let E and F be two Banach spaces $A : D(A) \subset E \rightarrow F$ an unbounded operator of dense and closed domain then :

1)A surjectif

2)$\exists\gamma > 0 : \|v\|_F \leq \|A^*v\|_E \, \forall v \in D(A^*)$

3)$\ker(A) = \{0\}$ and $\mathrm{Im}(A^*)$ is closed.

Definition 4 A minimizing sequence of criterion J on the set K is a sequence $(u_n)_{n\in IN} \subset K$ such that $\lim\limits_{n\rightarrow+\infty} J(u_n) = \inf\limits_{v\in K} J(v)$

Theorem 4 (Fubini) Let f be a continuous function on $[a, b] \times [c, d]$ with values in $\mathbb{C}$. Then

$$\int_a^b \left(\int_c^d f(x, y)dy\right) dx = \int_c^d \left(\int_a^b f(x, y)dx\right) dy.$$

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